Find all positive integers n for which

$$n^2+45$$

is a perfect square.

First we test some small values:

$n=0$ NO 45

$n=1$ NO 46

$n=2$ yes 49

$n=3$ NO $45+9=54$

$n=4$ NO $45+16=61$

$n=5$ NO $45+25=70$

$n=6$ yes

$$\begin{array}{r} 145 \\ 36 \\ \hline 81 \end{array}$$

$n=7$ NO

$$\begin{array}{r} 145 \\ 49 \\ \hline 94 \end{array}$$

$n=8$ NO

$$\begin{array}{r} 45 \\ 64 \\ \hline 109 \end{array}$$

$n=9$ NO

$$\begin{array}{r} 45 \\ 81 \\ \hline 126 \end{array}$$

$n=10$ NO 145

$n=11$ no $45+121=156$

Now we try an algebraic approach:

$$n^2 + 45 = k^2 \qquad (n, k \in \mathbb{Z})$$

$$\Rightarrow k^2 - n^2 = 45 \qquad \text{[rearrange]}$$

$$\Rightarrow (k-n)(k+n) = 45 \qquad \text{[difference of squares]}$$

What are the factors of 45?

1, 3, 5, 9, 15, 45

So obviously $k-n$ & $k+n$ must be one of those three pairs, and $k-n < k+n$

Case 1 $k-n = 1$, $k+n = 45$] solve simultaneously

add

$$\Rightarrow 2k = 46$$

$$\Rightarrow k = 23$$

$$\Rightarrow \boxed{n = 22}$$

Since it wasn't one of our earlier guesses, it can't hurt to double check it

Check: $22^2 + 45 = 529 = 23$ ✓

Case 2 $k-n = 3$, $k+n = 15$] solve simultaneously

add $\Rightarrow 2k = 18$

$\Rightarrow k = 9$

$\Rightarrow \boxed{n = 6}$

Yes - From tests above

Case 3 $\left.\begin{array}{l} k - n = 5 \\ k + n = 9 \end{array}\right\}$ Solve simultaneously

add

$\Rightarrow 2k = 14$

$\Rightarrow k = 7$

$\Rightarrow \boxed{n = 2}$

Yes - From tests above

So our solution is

$$\boxed{n = 2, 6, 22}$$