In a geometric sequence of real numbers, the sum of the first two terms is 7 and the sum of the first six terms is 91. What is the sum of the first four terms?

Our sequence is

$$a,\ ar,\ ar^2,\ \ldots$$

We have $a + ar = 7$ ①

and $a + \cdots + ar^5 = 91$ ②

So ① $\Rightarrow a(1+r) = 7 \Rightarrow a = \dfrac{7}{1+r}$

and ② $\Rightarrow \dfrac{a(1-r^6)}{1-r} = 91$ [sum of geometric series]

We have $1 - r^6 = 1^3 - (r^2)^3$

$= (1-r^2)(1+r^2+r^4)$

Now ② ÷ ①: $\dfrac{(1-r^6)}{(1+r)(1-r)} = \dfrac{91}{7}$

$\Rightarrow \dfrac{(1-r^2)(1+r^2+r^4)}{(1-r^2)} = 13$

$\Rightarrow r^4 + r^2 + 1 = 13$

$\Rightarrow r^4 + r^2 - 12 = 0$ which is a quadratic in r^2

$\Rightarrow (r^2-3)(r^2+4) = 0$

$\Rightarrow r^2 = 3$ or $r^2 = -4$

We must have $r^2 = 3$ since the series is real.

So the sum of the First Four terms is

$(1+r+r^2+r^3)a = [(1+r) + r^2(1+r)] \dfrac{7}{1+r}$

[Using ①]

$= 7(1+r^2) = 7\times(1+3)$

$= 28$