Let $a_1, a_2, \ldots$ be a sequence with the Following properties:

① $a_1 = 1$

② $a_{2n} = n \cdot a_n$ for any positive integer n

What is the value of $a_{2^{100}}$?

We have $a_1 = 1$

$a_2 = a_{2\times 1} = 1 \times a_1 = 1 = 2^0$

$a_4 = a_{2\times 2} = 2 \times a_2 = 2 = 2^1$

$a_8 = a_{2\times 4} = 4 \times a_4 = 8 = 2^3$

$a_{16} = a_{2\times 8} = 8 \times a_8 = 64 = 2^6$

$a_{32} = a_{2\times 16} = 16 \times a_{16} = 2^4 \cdot 2^6 = 2^{10}$

$a_{64} = a_{2\times 32} = 32 \times a_{32} = 2^5 \cdot 2^{10} = 2^{15}$

Pattern? a_{2^n} is 2 to the power

Of the $(n-1)^{th}$ triangular number, or

$$a_{2^n} = 2^{\frac{(n-1)n}{2}}$$

Proof by induction: know it's true for $n \leq 6$. Assume true for $n = k \in \mathbb{N}$

Then

$$a_{2^{(k+1)}} = a_{2 \cdot 2^k}$$

$$= 2^k a_{2^k}$$

$$= 2^k \cdot 2^{\frac{(k-1)k}{2}}$$

$$= 2^{\frac{2k + k^2 - k}{2}}$$

$$= 2^{\frac{k^2+k}{2}}$$

$$= 2^{\frac{k(k+1)}{2}}$$

which is 2 to the power of the k^{th} triangular number. Hence the statement is true by mathematical induction.

Now, $n=100 \Rightarrow a_{2^n} = 2^{\frac{99 \times 100}{2}}$

$$= 2^{4950}$$