C^*-algebras

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MATH235 Project

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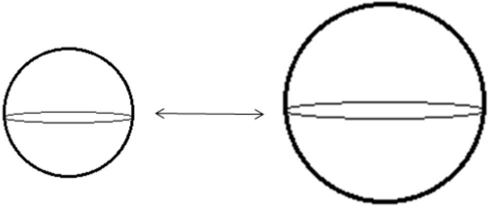
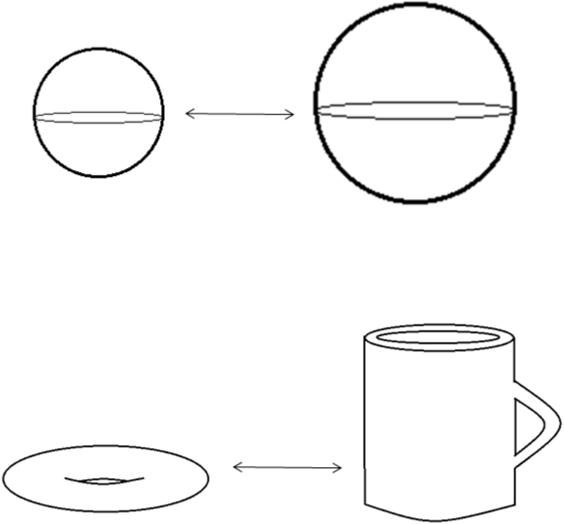
Topology: the study of shape and space

What does it mean for two spaces to be the same?

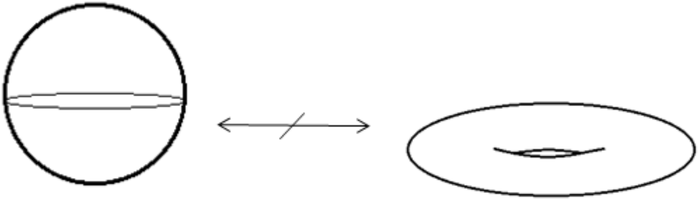
Homeomorphism: The idea of topological equivalence

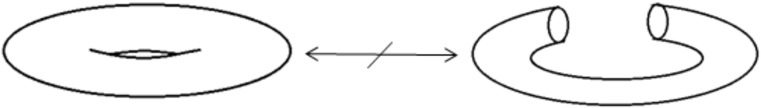


Homeomorphism: The idea of topological equivalence



Not equivalent





How we classify spaces: Invariants

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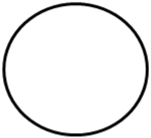
Compactness



Hausdorff

Generalising the complex numbers: What is a C*-algebra?

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Definition

A C*-algebra is a Banach algebra A with an involution satisfying the C*-identity $||a^*a|| = ||a||^2$ for all $a \in A$.

Examples

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n by n matrices with entries in $\mathbb{C}$: $M_n(\mathbb{C})$

$$\text{Involution: } M^* = \overline{M}^t$$

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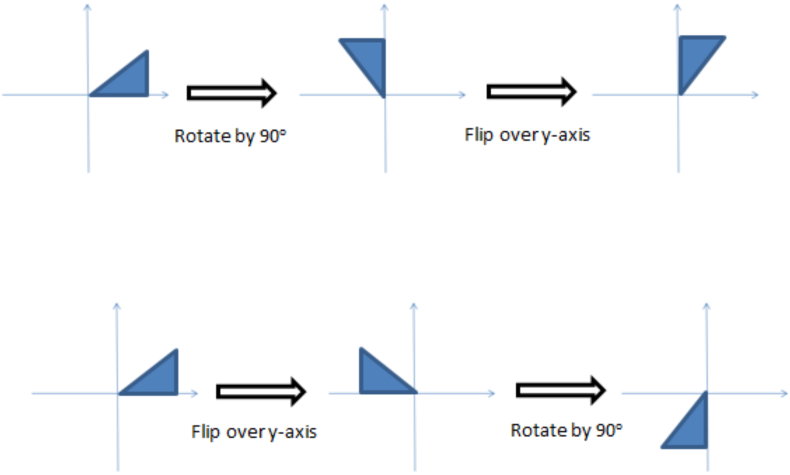
$M_n(\mathbb{C})$ has an *identity*, the matrix I_n, but it is *not commutative*.

$M_n(\mathbb{C})$ is non-commutative



Example

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Non-commutative since multiplication is composition of operators
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Classification: **Every** $C*$-algebra is isomorphic to a norm-closed, *-closed subalgebra of $B(H)$ for some Hilbert space H.

$$M_n(\mathbb{C}) = B(\mathbb{C}^n)$$

Example

The space of continuous, complex-valued functions on a compact Hausdorff space, $C(X)$

Involution: $f^*(x) = \overline{f(x)}$

Norm: $||f||_\infty = \sup_{x \in X} |f(x)|$

Multiplication: $(fg)(x) = f(x)g(x)$

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Observe that $\mathbb{C} = C(\text{pt})$, the continuous functions on a point. Which other C*-algebras also have this form?

The Gelfand Naimark Theorem

Theorem

Let A be a commutative C-algebra with 1. Then A is isomorphic to $C(X)$ for some compact, Hausdorff topological space X.*

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Let A be a commutative C^*-algebra with 1. Then A is isomorphic to $C(X)$ for some compact, Hausdorff topological space X.

Note: X is the space of all non-zero homomorphisms from A to $\mathbb{C}$ and the isomorphism in the theorem is given by the Gelfand transform:

$$\widehat{\ }: A \to C(X)$$
$$\widehat{a}(\phi) = \phi(a)$$

Consequences of the Gelfand-Naimark Theorem

$$X \xrightarrow{\phi} Y$$

$$\downarrow$$

$$C(X) \xleftarrow[\hat{\phi}]{} C(Y)$$

$$\hat{\phi}(f) = f \circ \phi$$