Zappa-Szep Products of Quantum Groups

March 10, 2016

Making New Groups: The Zappa-Szep Product

If we have two groups G and H and want to make a new one, how do we do it?

The Direct Product

Multiplication is $(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, h_1 h_2)$

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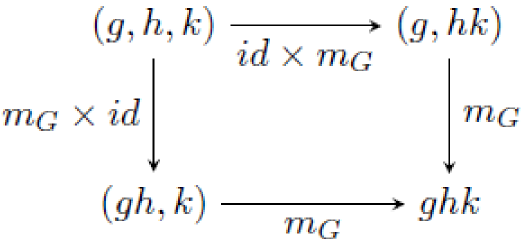
The (External) Zappa-Szep Product

Multiplication is $(g_1, h_1) \cdot (g_2, h_2) = (g_1 \alpha(h_1, g_2), \beta(h_1, g_2) h_2)$

where $\alpha : H \times G \to G, \beta : H \times G \to H$ satisfy certain properties

A Closer Look at Associativity in a Group

$(gh)k = g(hk)$ for all $g, h, k \in G$



Some Examples of C*-algebras

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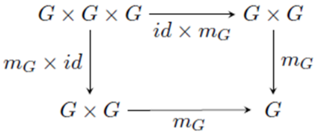
- **Continuous Complex-Valued Functions on a Compact Hausdorff Space**

$$\text{Involution: } f^*(x) = \overline{f(x)}$$
$$\text{Norm: } ||f||_\infty = \sup_{x \in X} |f(x)|$$
$$\text{Multiplication: } (fg)(x) = f(x)g(x)$$

Coassociativity in $C(G)$

Another way we could describe associativity in a group:



Multiplication: $m_G : G \times G \to G$

Coassociativity in $C(G)$

Another way we could describe associativity in a group:

$$\begin{array}{ccc} G \times G \times G & \xrightarrow[id \times m_G]{} & G \times G \\ {\scriptstyle m_G \times id}\downarrow & & \downarrow{\scriptstyle m_G} \\ G \times G & \xrightarrow[m_G]{} & G \end{array}$$

Multiplication: $m_G : G \times G \to G$
After applying the Gelfand transform:

$$\begin{array}{ccc} C(G) & \xrightarrow[\Phi_{C(G)}]{} & C(G) \otimes C(G) \\ {\scriptstyle \Phi_{C(G)}}\downarrow & & \downarrow{\scriptstyle id \otimes \Phi_{C(G)}} \\ C(G) \otimes C(G) & \xrightarrow[\Phi_{C(G)} \otimes id]{} & C(G) \otimes C(G) \otimes C(G) \end{array}$$

Comultiplication: $\Phi : C(G) \to C(G) \otimes C(G)$

What is a Quantum Group?

A pair (A, Φ) of a non-commutative **C*-algebra** A with an identity, and a **comultiplication** Φ that is coassociative:

$$(\Phi_A \otimes id) \circ \Phi_A = (id \otimes \Phi_A) \circ \Phi_A$$

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My project: Quantum group + Quantum group = Another quantum group?

Suppose H, K are groups and $\alpha : K \times H \to H$, $\beta : K \times H \to K$ are mappings satisfying:

- For all $k \in K$, $h \to \alpha(k, h)$ is a bijection
- For all $h \in H$, $k \to \beta(k, h)$ is a bijection
- For all $h \in H$, $\alpha(e, h) = h$
- For all $k \in K$, $\beta(k, e) = k$
- $\alpha(k_1 k_2, h) = \alpha(k_1, \alpha(k_2, h))$ for all $k_1, k_2 \in K, h \in H$
- $\beta(k_1 k_2, h) = \beta(k_1, \alpha(k_2, h))\beta(k_2, h)$ for all $k_1, k_2 \in K, h \in H$
- $\alpha(k, h_1 h_2) = \alpha(k, h_1)\alpha(\beta(k, h_1), h_2)$ for all $k \in K, h_1, h_2 \in H$
- $\beta(k, h_1 h_2) = \beta(\beta(k, h_1), h_2)$ for all $k \in K, h_1, h_2 \in H$

The **external Zappa-Szep product** of groups H, K is the set
$H \times K$ with multiplication
$(h_1, k_1)(h_2, k_2) = (h_1 \alpha(k_1, h_2), \beta(k_1, h_2)k_2)$ and inversion
$(h, k)^{-1} = (\alpha(k^{-1}, h^{-1}), \beta(k^{-1}, h^{-1}))$.

Theorem

Let A be a commutative C-algebra with 1. Then A is isomorphic to $C(X)$ for some compact, Hausdorff topological space X.*

Note: X is the space of all non-zero homomorphisms from A to $\mathbb{C}$ and the isomorphism in the theorem is given by the Gelfand transform:

$$\widehat{\ }: A \to C(X)$$
$$\widehat{a}(\phi) = \phi(a)$$

Theorem

Let (A, Φ_A) *and* (B, Φ_B) *be compact quantum groups. Suppose* $P : A \otimes B \to B \otimes A$ *is an isomorphism satisfying:*

$$(\Phi_B \otimes id_A)P = (id_B \otimes P)(P \otimes id_B)(id_A \otimes \Phi_B)$$
$$(id_B \otimes \Phi_A)P = (P \otimes id_A)(id_A \otimes P)(\Phi_A \otimes id_B)$$

Let $\Delta : A \otimes B \to A \otimes B \otimes A \otimes B$ *be given by*

$$\Delta = (id_A \otimes P \otimes id_B)(\Phi_A \otimes \Phi_B)$$

Then $(A \otimes B, \Delta)$ *is a compact quantum group.*