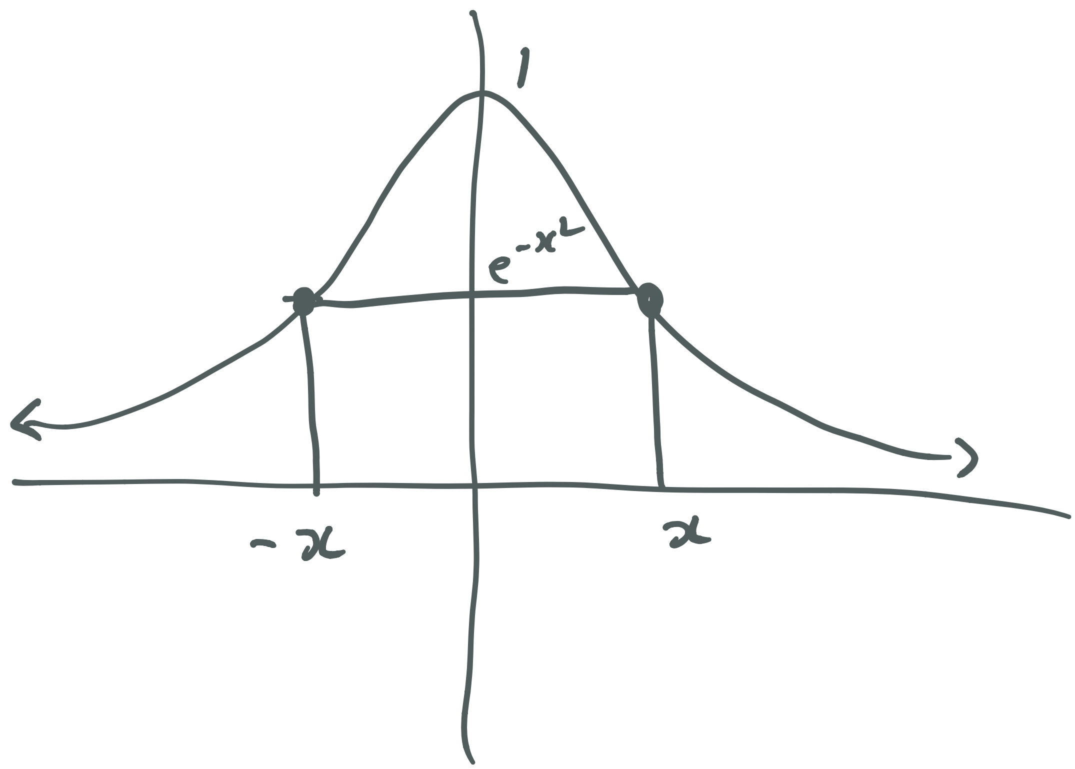
Determine the area of the largest rectangle with 1 side on x-axis and 2 vertices on curve $y = e^{-x^2}$. Express answer in simplest form.



$$\text{Area} = \text{base} \times \text{height}$$
$$= (2x) \times e^{-x^2}$$
$$= 2xe^{-x^2}$$

We want to <u>maximise</u> area

Define function $f(x) := 2x e^{-x^2}$

$\therefore \quad f'(x) = 2e^{-x^2} + 2x(-2x)e^{-x^2}$

$= e^{-x^2}\left[2 - 4x^2\right]$

and $\quad f''(x) = -2xe^{-x}\left[2-4x^2\right]$

$+ e^{-x^2}\left[-8x\right]$

$= e^{-x^2}\left[-8x - 4x + 8x^3\right]$

$= e^{-x^2}\left[-12x + 8x^3\right]$

Now, extreme pts occur when $f'(x) = 0$

ie $\quad e^{-x^2}\left[2-4x^2\right] = 0$

$\Rightarrow \quad 2 - 4x^2 = 0$

$\Rightarrow \quad 4x^2 = 2$

$\Rightarrow \quad x^2 = \frac{1}{2}$

$\Rightarrow \quad x = \pm\frac{1}{\sqrt{2}}$

When $x = \frac{1}{\sqrt{2}}$, $f''(x) < 0 \Rightarrow$ max

When $x = \frac{-1}{\sqrt{2}}$, $f''(x) > 0 \Rightarrow$ Min

So the maximum area occurs when

$$x = \frac{1}{\sqrt{2}}$$

$$\therefore \text{Area} = f\left(\frac{1}{\sqrt{2}}\right)$$

$$= 2\left(\frac{1}{\sqrt{2}}\right) e^{-\left(\frac{1}{\sqrt{2}}\right)^2}$$

$$= \boxed{\sqrt{2}\, e^{-\frac{1}{2}}} \text{ units}^2$$