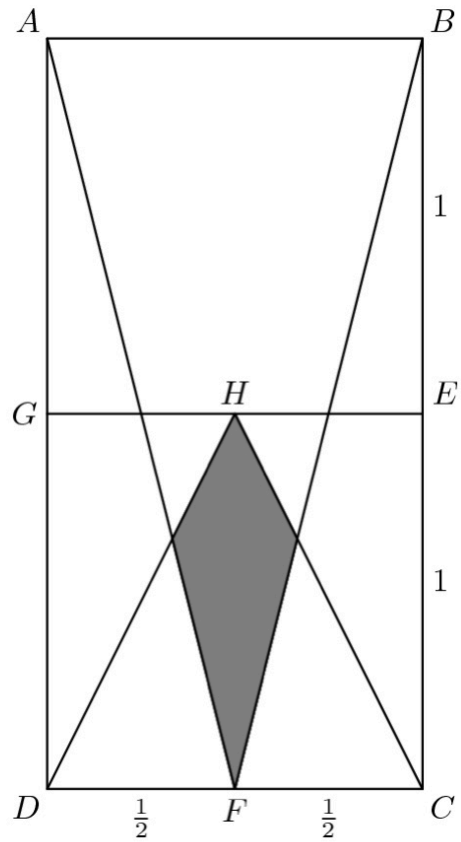
In rectangle $ABCD$, $AB = 1$, $BC = 2$, and points E, F, and G are midpoints of $\overline{BC}$, $\overline{CD}$, and $\overline{AD}$, respectively. Point H is the midpoint of $\overline{GE}$. What is the area of the shaded region?



Consider the rectangle as part of the coordinate plane so that D is at the origin. We have

$$H = \left(\tfrac{1}{2}, 1\right)$$
$$F = \left(\tfrac{1}{2}, 0\right)$$
$$A = (0, 2)$$

So the line joining D and H is given by

$$y = 2x$$

(by inspection if you wanted to calculate it, set $y = mx + b$ and sub the points D and H into the equation)

Now consider the line joining A and F.

Suppose it is $y = mx + b$.

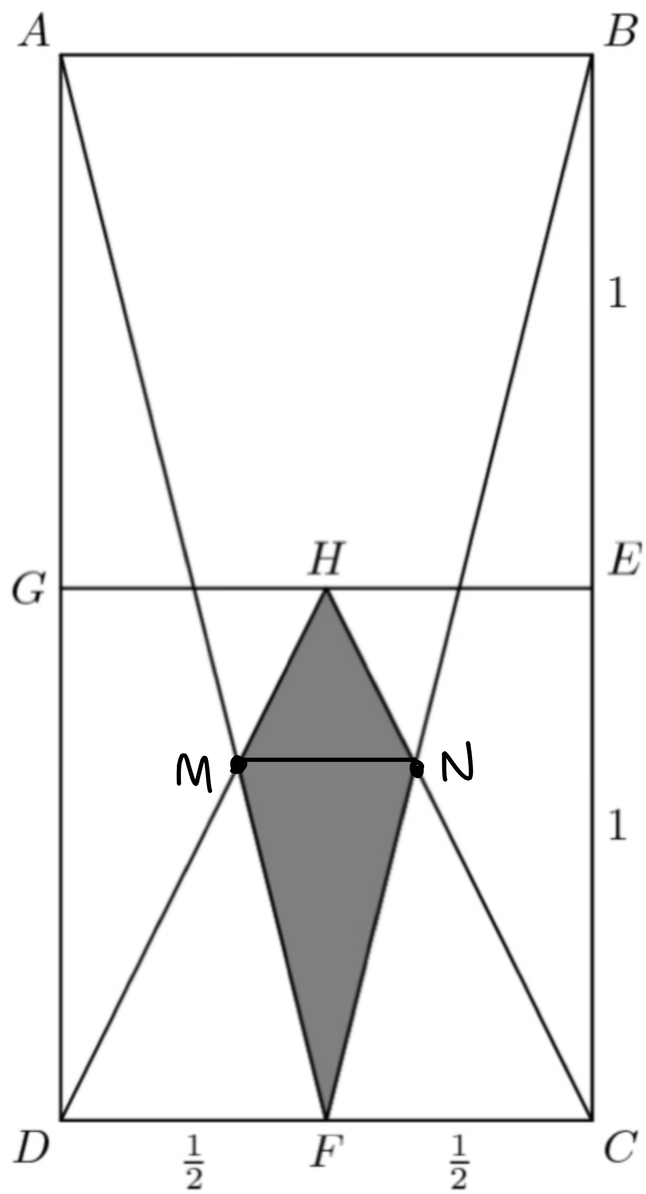
At A: $2 = b$

At F: $0 = \frac{1}{2}m + 2 \Rightarrow -2 = \frac{1}{2}m$

$\Rightarrow m = -4$

So $y = -4x + 2$

These two lines intersect at the point M on the diagram:



We have $2x = -4x + 2$

$$\Rightarrow \quad 6x = 2$$

$$\Rightarrow \quad x = 1/3$$

So $M = \left(\frac{1}{3}, \frac{2}{3}\right)$

By symmetry, the x-coordinate of N is

$$\frac{1}{2} + \underbrace{\left(\frac{1}{2} - \frac{1}{3}\right)}_{\substack{\text{dist from} \\ x\text{-coord of } M \\ \text{to } x\text{-coord} \\ \text{of } F}} \circledast = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$$

N is at the same height as M so its y-coordinate is also $2/3$. Hence $N = \left(\frac{2}{3}, \frac{2}{3}\right)$

Consider the triangle ΔMNF. It has height $\frac{2}{3}$ and base $\frac{1}{3}$ [this is found by twice the distance $\circledast$ above]. Hence the area of the triangle is $\frac{1}{2} \times \frac{1}{3} \times \frac{2}{3} = \frac{1}{9}$ units^2

Now consider the triangle ΔMNH. It has base $\frac{1}{3}$ [as in the previous triangle] and height $\frac{1}{3}$ [found by $1 - \frac{2}{3}$, the height at H minus the height at M and N]

So its area is $\frac{1}{2} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{18}$.

In conclusion, the total area of the shaded region is

$$\frac{1}{9} + \frac{1}{18} = \frac{2}{18} + \frac{1}{18}$$

$$= \frac{3}{18}$$

$$= \boxed{\frac{1}{6} \text{ units}^2}$$