

Write $\frac{1}{2}$ in base 3

$$0.1_3 = 1 \times \frac{1}{3} < \frac{1}{2}$$

$$0.2_3 = 2 \times \frac{2}{3} > \frac{1}{2}$$

\therefore 1st dp is 1

$$0.11_3 = \frac{1}{3} + \frac{1}{9} = \frac{3+1}{9} = \frac{4}{9} < \frac{1}{2}$$

$$0.12_3 = \frac{1}{3} + \frac{2}{9} = \frac{3+2}{9} = \frac{5}{9} > \frac{1}{2}$$

\therefore 2nd dp is a 1

We've found a repetition, so guess that

$$\frac{1}{2} = 0.\dot{1}_3$$

Check:

$$0.\dot{1}_3 = \sum_{k=1}^{k=\infty} \frac{1}{3^k}$$

$$= \frac{\frac{1}{3}}{1 - \frac{1}{3}}$$

$$= \frac{\frac{1}{3}}{\frac{2}{3}}$$

$$= \frac{2}{3} \quad \text{as expected}$$

Find $\frac{5}{6}$ as a base 5 decimal

$$0.4_5 = \frac{4}{5} < \frac{5}{6} \quad \text{so the first dp is a 4}$$

$$0.40_5 = \frac{4}{5} < \frac{5}{6}$$

$$0.41_5 = \frac{4}{5} + \frac{1}{25} > \frac{5}{6} \quad \text{so the second dp is a 0.}$$

$$0.404_5 = \frac{4}{5} + \frac{4}{125} < \frac{5}{6} \quad \text{so the third dp}$$

is a 4.

$$0.4041_5 > \frac{5}{6} \quad \text{so the fourth dp is a 0.}$$

We notice a pattern emerging and guess

$$\text{that } \frac{5}{6} = 0.\overline{40}_5$$

Check :

$$0.\overline{40}_5 = 4 \sum_{k=0}^{\infty} \frac{1}{5^{2k+1}}$$

$$= \frac{4 \left(\frac{1}{5} \right)}{1 - \frac{1}{25}}$$

$$= \frac{4/5}{24/25}$$

$$= \frac{4}{5} \cdot \frac{25}{24}$$

$$= \frac{5}{6} \text{ as expected}$$