Show that if m is a positive rational number then $m + \frac{1}{m}$ is an integer only if $m=1$

$$m + \frac{1}{m} = n \quad \text{where } n \in \mathbb{Z}$$

$$\Rightarrow \quad m^2 + 1 = nm$$

$$\Rightarrow \quad m^2 - nm + 1 = 0$$

$$\Rightarrow \quad m = \frac{n \pm \sqrt{n^2 - 4}}{}$$

Since m is a rational number, we know that n^2-4 is a perfect square.

Firstly, $n^2-4 < n^2$

Secondly, $(n-1)^2 = n^2-2n+1$ *

$< n^2-4$ when $n \geq 3$

Hence $(n-1)^2 < n^2-4 < n^2$

This is a contradiction since we don't have any square numbers between consecutive squares.

So check $n=1$, $n=2$:

If $n=1$ then $n^2-4<0$ so no real roots exist.

If $n=2$ then $n^2-4=0$ so the only solution is

$$m = \frac{2 \pm 0}{2} = 1$$

* $n^2-2n+1 = n^2-(2n-1)$

$< n^2 - 4$ when $2n-1 > 4$

$\Rightarrow 2n > 5$

$\Rightarrow n > 2.5$

$\Rightarrow n \geq 3$ since $n \in \mathbb{N}$