

Find $x^4 + \frac{1}{x^4}$ if $x - \frac{1}{x} = 5$

We have

$$\begin{aligned}\left(x^2 + \frac{1}{x^2}\right)^2 &= x^4 + 2x^2 \frac{1}{x^2} + \frac{1}{x^4} \\ &= x^4 + 2 + \frac{1}{x^4}\end{aligned}$$

also

$$\begin{aligned}\left(x - \frac{1}{x}\right)^2 &= x^2 - 2x \frac{1}{x} + \frac{1}{x^2} \\ &= x^2 - 2 + \frac{1}{x^2}\end{aligned}$$

So

$$x^4 + \frac{1}{x^4} = \left(x^2 + \frac{1}{x^2}\right)^2 - 2$$

$$\begin{aligned} &= \left[\left(x - \frac{1}{x} \right)^2 + 2 \right]^2 - 2 \\ &= \left[5^2 + 2 \right]^2 - 2 \\ &= \left[27 \right]^2 - 2 \\ &= \boxed{727} \end{aligned}$$

Find all possible values of $x^3 + \frac{1}{x^3}$
given that $x^2 + \frac{1}{x^2} = 7$

$$\begin{aligned} x^3 + \frac{1}{x^3} &= \left(x + \frac{1}{x} \right) \left(x^2 - x \frac{1}{x} + \frac{1}{x^2} \right) \\ &= \left(x + \frac{1}{x} \right) \left(x^2 + \frac{1}{x} - 1 \right) \\ &= \left(x + \frac{1}{x} \right) (6) \end{aligned}$$

$$\begin{aligned}\text{Now } \left(x + \frac{1}{x}\right)^2 &= x^2 + 2 + \frac{1}{x^2} \\ &= 7 + 2 \\ &= 9\end{aligned}$$

$$\text{So } x + \frac{1}{x} = 3 \text{ or } -3$$

$$\begin{aligned}\text{So } x^3 + \frac{1}{x^3} &= 6 \times 3 = 18 \\ \text{or } 6x - 3 &= -18\end{aligned}$$