Find $x^4 + \frac{1}{x^4}$ if $x - \frac{1}{x} = 5$

We have

$$\left(x^2 + \frac{1}{x^2}\right)^2 = x^4 + 2x^2\frac{1}{x^2} + \frac{1}{x^4}$$

$$= x^4 + 2 + \frac{1}{x^4}$$

also

$$\left(x - \frac{1}{x}\right)^2 = x^2 - 2x\frac{1}{x} + \frac{1}{x^2}$$

$$= x^2 - 2 + \frac{1}{x^2}$$

So

$$x^4 + \frac{1}{x^4} = \left(x^2 + \frac{1}{x^2}\right)^2 - 2$$

$$= \left[\left(x - \frac{1}{x}\right)^2 + 2\right]^2 - 2$$

$$= \left[5^2 + 2\right]^2 - 2$$

$$= [27]^2 - 2$$

$$= 727$$

Find all possible values of $x^3 + \frac{1}{x^3}$ given that $x^2 + \frac{1}{x^2} = 7$

$$x^3 + \frac{1}{x^3} = \left(x + \frac{1}{x}\right)\left(x^2 - x\frac{1}{x} + \frac{1}{x^2}\right)$$

$$= \left(x + \frac{1}{x}\right)\left(x^2 + \frac{1}{x} - 1\right)$$

$$= \left(x + \frac{1}{x}\right)(6)$$

Now $\left(x + \frac{1}{x}\right)^2 = x^2 + 2 + \frac{1}{x^2}$

$= 7 + 2$

$= 9$

So $x + \frac{1}{x} = 3$ or -3

So $x^3 + \frac{1}{x^3} = 6 \times 3 = 18$

or $6 \times -3 = -18$