The zeroes of the function $f(x) = x^2 - ax + 2a$ are integers. What is the sum of the possible values of a?

(A) 7 **(B)** 8 **(C)** 16 **(D)** 17 **(E)** 18

The Function has two zeroes (since it's quadratic), α and $\beta \in \mathbb{Z}$.

We have

$$\alpha + \beta = \frac{-(-a)}{1} = a$$

So $a \in \mathbb{Z}$ also.

Now, the zeroes are given by the quadratic Formula

$$\frac{a \pm \sqrt{a^2 - 8a}}{2}$$

hence $a^2 - 8a = k^2$ [a perfect square] $(k \in \mathbb{Z})$

Complete the square:

$$(a-4)^2 - 16 = k^2$$

$$\Rightarrow (a-4)^2 - k^2 = 16$$

$\Rightarrow (a-4+k)(a-4-k) = 16$

So the possible factors on the LHS are

$(1,16), (2,8), (4,4), (-1,-16), (-2,-8), (-4,-4)$

Try them all:

$\begin{cases} a-4+k=16 \\ a-4-k=1 \end{cases} \overset{\text{add}}{\Rightarrow} 2a-8=17$

$\Rightarrow 2a = 25$ No $(a \notin \mathbb{Z})$

$\begin{cases} a-4+k=8 \\ a-4-k=2 \end{cases} \overset{\text{add}}{\Rightarrow} 2a-8=10$

$\Rightarrow 2a=18$

$\Rightarrow a=9$ Yes

$\begin{cases} a-4+k=4 \\ a-4-k=4 \end{cases} \overset{\text{add}}{\Rightarrow} 2a-8=8$

$\Rightarrow 2a=16$

$\Rightarrow a=8$ Yes

$\begin{cases} a-4+k=-1 \\ a-4-k=-16 \end{cases} \overset{\text{add}}{\Rightarrow} 2a-8=-17$

$\Rightarrow 2a=-9$ No $(a \notin \mathbb{Z})$

$$\begin{cases} a-4+k = -2 \\ a-4-k = -8 \end{cases} \overset{\text{add}}{\Rightarrow} 2a-8 = -10$$

$$\Rightarrow 2a = -2$$

$$\Rightarrow a = -1 \qquad \text{yes}$$

$$\begin{cases} a-4+k = -4 \\ a-4-k = -4 \end{cases} \overset{\text{add}}{\Rightarrow} 2a-8 = -8 \qquad \text{yes}$$

$$\Rightarrow a = 0$$

So the possible values of a are

$9, 8, -1, 0$

and their sum is $9+8-1+0 = \boxed{16}$