THE PREDATOR PREY EQUATIONS: MODELLING TUNA AND LION POPULATIONS

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1. Introduction

In this paper, we use methods of difference and differential equations to model the effect of predation on tuna and lion populations using predator-prey equations. In the finite differences section, we derive a system of difference equations and find its fixed points analytically. In the differential equations section, we derive a system of ODEs and solve it numerically. We aim to analyse the trends in population of both species using a variety of initial conditions.

2. Assumptions

In order to simplify the model, we do not account for many factors that affect the animal populations in the real world, such as self-competition, competition with other species, limited availability of food, etc. The following assumptions are used throughout the paper:

(1) If there are no tunas present, then the lion population increases without bound.

(2) If there are no lions present, then the tuna population decreases to extinction.

(3) There are no other species that are eaten by the tunas, and no other species that eat the lions.

(4) As the population of tuna increases, the lion population decreases proportional to both the tuna and lion populations.

(5) As the population of lions increases, the tuna population increases proportional to both the tuna and lion populations.

(6) As the population of tuna decreases, the lion population increases proportional to both the tuna and lion populations.

(7) As the population of lions decreases, the tuna population decreases proportional to both the tuna and lion populations.

3. A System of Difference Equations

We derive a system of difference equations based on the above assumptions, closely following the exposition in [1].

Let x_t denote the size of the lion population at time t, and y_t denote the size of the tuna population at time t. We note that $x_t, y_t \geq 0 \quad \text{for all } t > 0$.

A suitable pair of word equations to model the populations would be

{ Lion population at time t } = { Lion population at time $t-1$ } + { Births } − { Natural deaths } − { Eaten by Tunas }

{ Tuna population at time t } = { Tuna population at time $t-1$ } + { Births } − { Deaths } + { Newborns due to availability of food }

We first model the population dynamics of each species separately:

Since the lion population increases to infinity in the absence of tunas, we use the model

$$x_{t+1} = K_1 x_t$$

where $K_1 > 1$ represents the growth rate. Note that we require $K_1 > 1$ because the solution to this difference equation is $x_t = K_1^t x_0$, which will increase without bound if and only if $K_1 > 1$.

Similarly, we can model the tuna population, in isolation, by the difference equation

$$y_{t+1} = K_2 y_t$$

where $0 < K_2 < 1$. This inequality follows similarly from the general solution, which is $y_t = K_2^t y_0$, since this will only approach 0 in the long term if $0 < K_2 < 1$.

Now, using the assumptions (4) - (7), we obtain the following non-linear difference equations:

$$x_{t+1} = K_1 x_t - K_3 x_t y_t \tag{3.1}$$

$$y_{t+1} = K_2 y_t + K_4 x_t y_t \tag{3.2}$$

Here, K_3 and K_4 are positive constants.

4. Solution

We now find the fixed points of this system: let X^* and Y^* be fixed points of 3.1 and 3.2. We obtain a pair of simultaneous equations that can be solved to find our fixed points:

$$X^* = K_1X^* - K_3X^*Y^* \tag{4.1}$$

$$Y^* = K_2Y^* + K_4X^*Y^* \tag{4.2}$$

Then from 4.1, we see that either $X^* = 0$ or $1 = K_1 - K_3Y^*$, which is equivalent to $Y^* = \frac{K_1-1}{K_3}$.

Substitute $X^* = 0$ into 4.2. We see that $Y^* = 0$ is the only solution, so one fixed point is $(X^*, Y^*) = (0, 0)$. The other fixed point is obtained by substituting $Y^* = \frac{K_1-1}{K_3}$ into 4.2. Then

$$\begin{aligned}\frac{K_1-1}{K_3} &= K_2\frac{K_1-1}{K_3} + K_4\frac{K_1-1}{K_3}X^* \\ X^* &= \frac{1-K_2}{K_4} \\ &> 0\end{aligned}$$

since $K_3 < 1$ and $K_4 > 0$. Thus, the other fixed point is $(X^*, Y^*) = (\frac{1-K_2}{K_4}, \frac{K_1-1}{K_3})$.

We now analyse the stability of the fixed points. Set $f(X^*, Y^*) = K_1X^* - K_3X^*Y^*$ and $g(X^*, Y^*) = K_2X^* - K_4X^*Y^*$. The Jacobian is

$$\begin{pmatrix} K_1 - K_3Y^* & -K_3X^* \\ K_3Y^* & K_3 + K_4X^* \end{pmatrix}$$

We first consider the trivial fixed point. At $(0, 0)$, this matrix is

$$\begin{pmatrix} K_1 & 0 \\ 0 & K_3 \end{pmatrix}$$

The eigenvalues are $\lambda_1 = K_1$ and $\lambda_2 = K_3$. For stability we need $|\lambda_i| < 1$ for $i = 1, 2$. We know that $K_3 < 1$ but $K_1 > 1$, so we conclude that the trivial fixed point is not stable.

Now we consider the non-trivial fixed point $(\frac{1-K_2}{K_4}, \frac{K_1-1}{K_3})$. Here, the Jacobian is

$$\begin{aligned} J &= \begin{pmatrix} K_1 - K_3\left(\frac{K_1-1}{K_3}\right) & -K_3\left(\frac{1-K_2}{K_4}\right) \\ K_4\left(\frac{K_1-1}{K_3}\right) & K_2 + K_4\left(\frac{1-K_2}{K_4}\right) \end{pmatrix} \\ &= \begin{pmatrix} 1 & -K_3\left(\frac{1-K_2}{K_4}\right) \\ K_4\left(\frac{K_1-1}{K_3}\right) & 1 \end{pmatrix}\end{aligned}$$

Now solve the characteristic polynomial to obtain the eigenvalues:

$$\begin{aligned}(1-\lambda)^2 - K_3K_4\left(\frac{1-K_2}{K_4}\right)\left(\frac{K_1-1}{K_3}\right) &= 0\\ \lambda^2 - 2\lambda + 1 - (-1+K_2)(K_1-1) &= 0\\ \lambda^2 - 2\lambda + K_1 + K_2 - K_1K_2 &= 0\end{aligned}$$

Using the quadratic formula:

$$\begin{aligned}\lambda &= \frac{2 \pm \sqrt{4 - 4(K_1 + K_2 - K_1K_2}}{2}\\ &= 1 \pm \sqrt{K_1 + K_2 - K_1K_2}\end{aligned}$$

It is therefore not possible to have $|\lambda_i| < 1$ for $i = 1, 2$, so the non-trivial fixed point is also unstable.

5. Interpretation

Both fixed points are unstable. This means that the populations do not approach a constant - over time, they oscillate.

6. A System of Differential Equations

In this section, we derive the Lotka-Volterra equations.

Let $x : [0, \infty) \to [0, \infty)$ represent the lion population as a function of time, and $y : [0, \infty) \to [0, \infty)$ represent the tuna population as a function of time.

We first model each population separately:

Using assumption (1), we have:

$$\frac{dx}{dt} = ax$$

where $a > 0$ represents the growth rate of the lion population.

Note that it is also possible to use a logistic model $\frac{dx}{dt} = ax - bx^2$, where $b > 0$, to represent a limited food supply and competition between the lions for limited resources. The equilibrium points of this model are $x = 0$ and $x = \frac{a}{b}$. This is more realistic but for simplicity, we shall use the model above.

Using assumption (2), we have:

$$\frac{dy}{dt} = -cy$$

where $c > 0$ represents the growth rate of the tuna population.

Again, we could use the model $\frac{dy}{dt} = -cy - dy^2$, where $d > 0$.

Now we model the interaction between the species. Using assumptions (4) - (7) we obtain the equations:

$$\frac{dx}{dt} = x(a - my) \qquad \text{where } m > 0 \tag{6.1}$$

$$\frac{dy}{dt} = y(-c + nx) \qquad \text{where } n > 0 \tag{6.2}$$

This system of differential equations is commonly known as the Lotka Volterra equations.

7. Solution

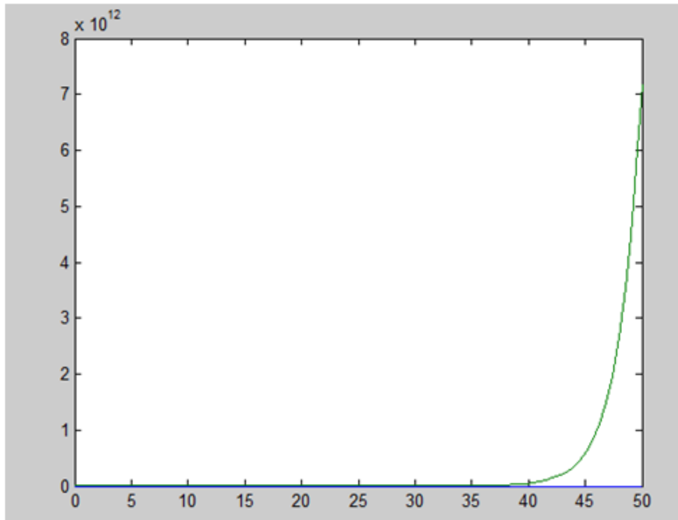
Since we cannot use the techniques taught in MATH111 to solve equations 6.1 and 6.2 analytically, we will set a, m, c, n equal to constants and solve the following system numerically:

$$\frac{dx}{dt} = x(0.5 - 0.01y)$$

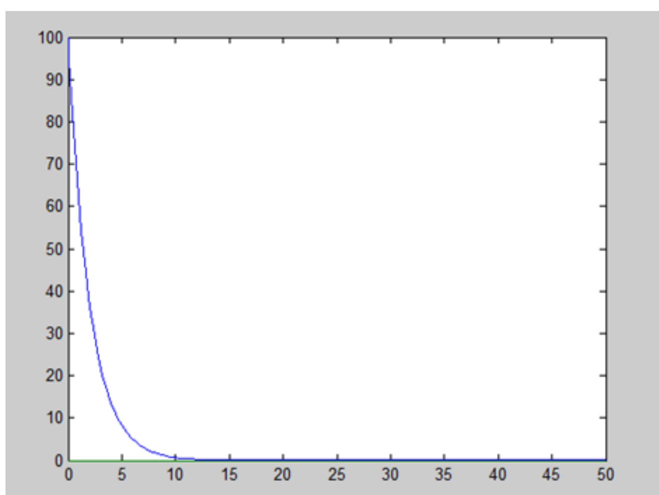
$$\frac{dy}{dt} = y(-0.5 + 0.01x)$$

See the Appendix for the MATLAB code used to produce the following plots. In each graph, the lion population is represented in green, and the tuna population is represented in blue. The x-axis represents time (measured in months), and the y-axis represents population.

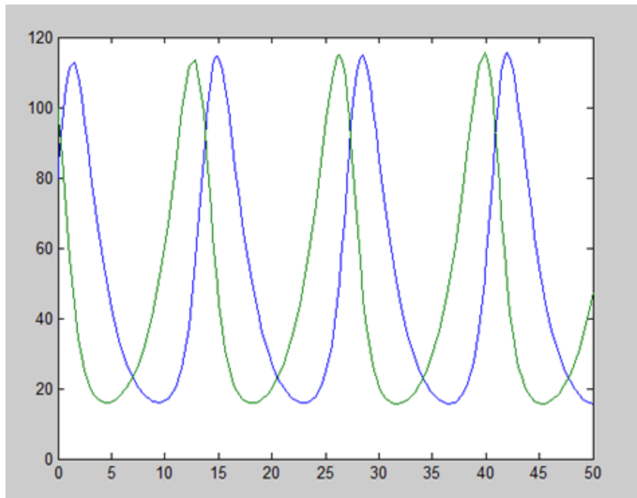
First, we examine the size of the lion population over time when no tunas are present. As expected from the assumptions, we see that the lion population grows without bound. The following graph is produced from the choice of initial conditions tunas = 0, lions = 100:



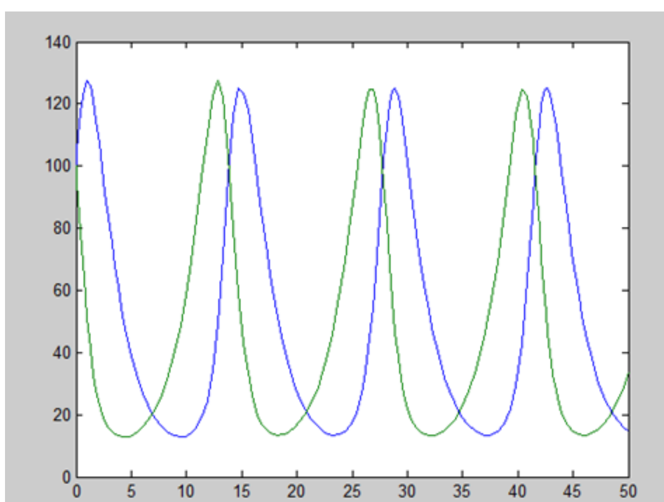
Now, we examine the size of the tuna population over time when no lions are present. Again, the results are as expected: the tuna population declines since no food is available. The following graph is produced from the choice of initial conditions tunas = 100, lions = 0:



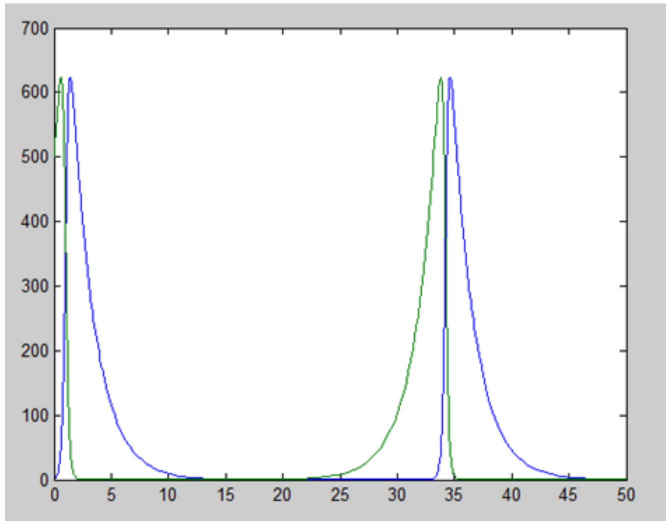
We will now let both initial populations be non-zero: set tunas = 80, lions = 100. The following graph is obtained:



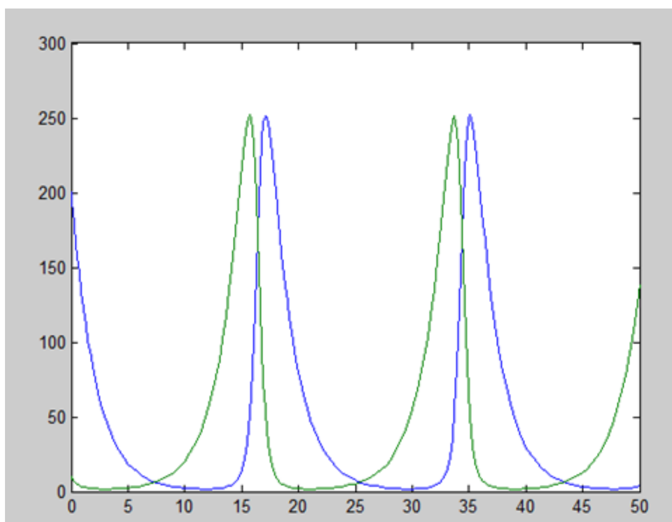
Similar behaviour is observed when we set tunas = 100, lions = 100:



Surprisingly, we see similar behaviour in the case where tunas = 2, lions = 100:



Finally, we observe that the graph of the solution created when tunas = 200, lions = 10 is, unexpectedly, similar:



8. Interpretation of Results

From these graphs it can be concluded that the populations of both predator and prey oscillate. This is a consequence of the assumptions. Initially, the lion population decreases due to the presence of tunas. Then the tuna population decreases since there are less lions. So the lion population increases due to the smaller amount of predators. Finally, the tuna population increases since more food is available. The cycle then repeats.

It seems counterintuitive that when a population is very small, it does not die out completely. Realistically, this may in fact happen due to lack of food, competition between members of the same species, disease, predation from other species, and so on. The model we used in this paper does not account for these factors.

9. Generalisations

The equations developed in this paper are useful in the modelling of the interaction of two species. We can generalise this to the modelling of the interaction between n species using the following system of differential equations, from [7]:

$$\frac{dx_i}{dt} = x_i\big(r_i + \sum_j a_{ij}x_j\big)$$

Here, x_i denotes the population density of the i^{th} species, r_i represents this species' growth rate, and the matrix $A = (a_{ij})$, called the interaction matrix, determines the interaction between the species.

A similar model can be used to analyse food chains: in such a structure, the first species is prey for the second species, the second species is prey for the third species, and so on. In [7], the following system of equations is used to model a food chain:

$$\begin{aligned}\frac{dx_1}{dt} &= x_1(r_1 - a_{11}x_1 - a_{12}x_2)\\ &\vdots\\ \frac{dx_i}{dt} &= x_i(-r_i + a_{i,i-1}x_{i-1} - a_{ii}x_i - ai, i + 1x_{j+1})\\ &\vdots\\ \frac{dx_n}{dt} &= x_n(-r_n + a_{n,n-1}x_{n-1} - a_{nn}x_n)\end{aligned}$$

with all r_i and a_{ij} positive.

10. Conclusion

Depending on the time period used in the modelling process, both the difference and differential equation models could be suitable ways to predict future population growth. If the time interval chosen is large (e.g. a year) then difference equations are more suitable; however, if the time interval is short (e.g. a month) then differential equations are more appropriate. Both models showed that the populations do not approach a constant size in the long term; rather, they oscillate over time.

11. Appendix

The following code, from [2], was used to solve the Lotka Volterra equations numerically:

```
clear;
to = 0;
tf = 50;
yo = [200 100];
[t y] = ode45('yprf',[to tf],yo);
plot(t,y(:,1),t,y(:,2))
```

```
function yprf =yprf(t,y)
yprf(1) = -.5*y(1) + .01*y(1)*y(2);
yprf(2) = .5*y(2) -.01*y(1)*y(2);
yprf = [yprf(1) yprf(2)]';
```

The first function solves and plots the ODEs defined by the second function based on given initial conditions.

References

[1] Thamwattana, N.: *Math111 Lecture Notes*, (2014)

[2] White: *http://www4.ncsu.edu/eos/users/w/white/www/white/ma302/less708.pdf*, [Accessed 10/10/14]

[3] Tseng, Z.S: *http://www.math.psu.edu/tseng/class/Math251/Notes-Predator-Prey.pdf*, [Accessed 10/10/14]

[4] Moler, C.: *https://www.mathworks.com.au/moler/exm/chapters/predprey.pdf*, [Accessed 10/10/14]

[5] Bryant, J., Moyer, J.: *http://www.resnet.wm.edu/ jxshix/math302/byrant-moyer2.pdf*, [Accessed 10/10/14]

[6] Montgomery, J.: *http://www.ssc.wisc.edu/ jmontgom/predatorprey.pdf*, [Accessed 10/10/14]

[7] Sternberg, S.: *http://www.math.harvard.edu/library/sternberg/slides/11809LV.pdf*, [Accessed 10/10/14]