AMSI

AMSI HIGHER ED

Zappa-Szep Products of Compact Quantum Groups

Penelope Drastik
Supervisors Nathan Brownlowe and Dave Robertson
University of Wollongong

Australian Government
Department of Education and Training

1 Introduction

In this report we review the definition of a quantum group and the external Zappa-Szep product, and present a theorem about Zappa-Szep products of quantum groups.

2 The Gelfand Naimark Theorem

Theorem 1 (The Gelfand Naimark Theorem) *If A is a commutative unital C^*-algebra then $A \cong C(\Delta)$ where Δ is the set of non-zero homomorphisms from A to $\mathbb{C}$.*

Note that the isomorphism is given by the Gelfand transform.

Suppose that $A \cong C(G)$ where G is a compact group. We know that associativity holds in G, that is, $(gh)k = g(hk)$ for all $g, h, k \in G$. This can be expressed as a commutative diagram:

$$\begin{array}{ccc} G \times G \times G & \xrightarrow[id \times m_G]{} & G \times G \\ {\scriptstyle m_G \times id}\downarrow & & \downarrow{\scriptstyle m_G} \\ G \times G & \xrightarrow[m_G]{} & G \end{array}$$

$$\begin{array}{ccc} (g,h,k) & \xrightarrow[id \times m_G]{} & (gh,k) \\ {\scriptstyle m_G \times id}\downarrow & & \downarrow{\scriptstyle m_G} \\ (g,hk) & \xrightarrow[m_G]{} & ghk \end{array}$$

Now, using properties of the Gelfand transform, we obtain the following diagram:

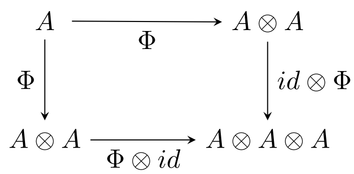
$$\begin{array}{ccc} C(G) & \xrightarrow[\Phi_{C(G)}]{} & C(G) \otimes C(G) \\ {\scriptstyle \Phi_{C(G)}}\downarrow & & \downarrow{\scriptstyle id \otimes \Phi_{C(G)}} \\ C(G) \otimes C(G) & \xrightarrow[\Phi_{C(G)} \otimes id]{} & C(G) \otimes C(G) \otimes C(G) \end{array}$$

In the diagram, $\Phi_{C(G)} : C(G) \to C(G) \otimes C(G)$ is given by $[\Phi(f)](g,h) = f(gh)$. We see that $(\Phi_{C(G)} \otimes id)\Phi_{C(G)} = (id \otimes \Phi_{C(G)})\Phi_{C(G)}$ holds. This will be important motivation for the definition of a quantum group.

3 Compact Quantum Groups

The motivation behind the definition of a compact group is to extend the result obtained in the previous section to non-commutative C*-algebras.

Definition 1 *$\Phi : A \to A \otimes A$ is a* ***coassociative comultiplication*** *on a unital C*-algebra A if it is a unital homomorphism satisfying $(\Phi \otimes id)\Phi = (id \otimes \Phi)\Phi$*



This is a reasonable definition to make since it generalises the result for commutative C*-algebras.

Another property of Φ is given in the following theorem:

Theorem 2 *[1] If G is a compact semi-group then it has cancellation if and only if the sets $\Phi(C(G))(1 \otimes C(G))$ and $\Phi(C(G))(C(G) \otimes 1)$ are dense in $C(G) \otimes C(G)$.*

We want a similar property to hold in our non-commutative generalisation:

Definition 2 *A* ***compact quantum group*** *is a pair (A, Φ) where A is a unital C*-algebra, Φ is a coassociative comultiplication, and the sets $\Phi(A)(1 \otimes A)$ and $\Phi(A)(A \otimes 1)$ are dense in $A \otimes A$.*

Using [1], we can characterise a compact group as a set G with an associative, continuous multiplication $m : G \times G \to G$ which satisfies the cancellation property. The definition of a quantum group has corresponding properties:

multiplication in $G \leftrightarrow$ coassociative comultiplication

cancellation in $G \leftrightarrow$ sets $\Phi(A)(1 \otimes A)$ and $\Phi(A)(A \otimes 1)$ are dense in $A \otimes A$

4 The Zappa-Szep Product

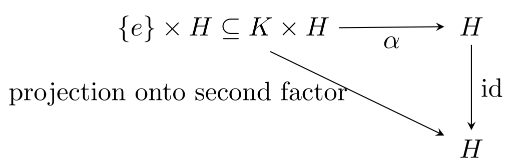
Definition 3 *Suppose* H, K *are groups and* $\alpha : K \times H \to H$, $\beta : K \times H \to K$ *are mappings satisfying:*

- *For all* $k \in K$, $h \to \alpha(k, h)$ *is a bijection*
- *For all* $h \in H$, $k \to \beta(k, h)$ *is a bijection*
- *For all* $h \in H$, $\alpha(e, h) = h$
- *For all* $k \in K$, $\beta(k, e) = k$
- $\alpha(k_1 k_2, h) = \alpha(k_1, \alpha(k_2, h))$ *for all* $k_1, k_2 \in K, h \in H$
- $\beta(k_1 k_2, h) = \beta(k_1, \alpha(k_2, h))\beta(k_2, h)$ *for all* $k_1, k_2 \in K, h \in H$
- $\alpha(k, h_1 h_2) = \alpha(k, h_1)\alpha(\beta(k, h_1), h_2)$ *for all* $k \in K, h_1, h_2 \in H$
- $\beta(k, h_1 h_2) = \beta(\beta(k, h_1), h_2)$ *for all* $k \in K, h_1, h_2 \in H$

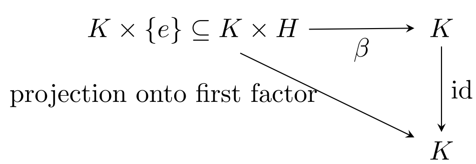
The ***external Zappa-Szep product*** *of groups* H, K *is the set* $H \times K$ *with multiplication* $(h_1, k_1)(h_2, k_2) = (h_1 \alpha(k_1, h_2), \beta(k_1, h_2) k_2)$ *and inversion* $(h, k)^{-1} = (\alpha(k^{-1}, h^{-1}), \beta(k^{-1}, h^{-1}))$.

We can express the above properties using commutative diagrams, which will be used to define the Zappa-Szep product of compact quantum groups.

For 3:

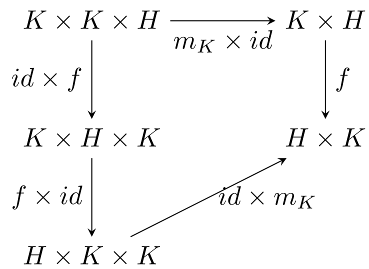


For 4:

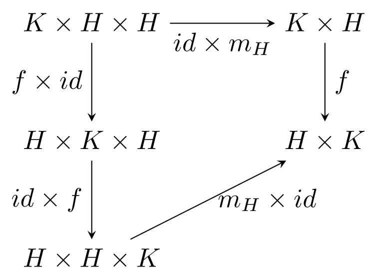


Define a mapping $f : K \times H \to H \times K$, where $f(k,h) = (\alpha(k,h), \beta(k,h))$.

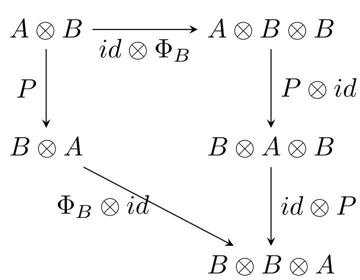
For 5 and 6:



For 7 and 8:



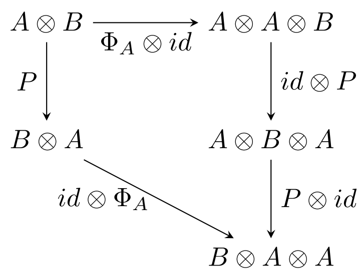
Now, suppose that H, K are compact and α, β are continuous (so f is continuous). We will apply the same idea that was used to obtain the definition of a quantum group to produce a commutative diagram giving information about quantum groups (A, Φ_A) and (B, Φ_B) that can be used to define their Zappa-Szep product. We assume that there exists an isomorphism $P : A \otimes B \to B \otimes A$ which corresponds to f in the previous diagrams.



Thus, we obtain

$(\Phi_B \otimes id_A)P = (id_B \otimes P)(P \otimes id_B)(id_A \otimes \Phi_B)$

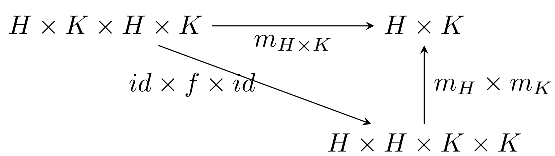
Similarly:



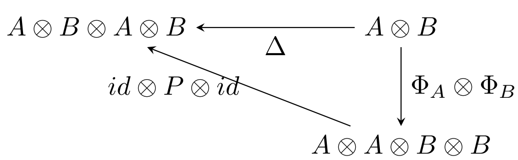
So:

$(id_B \otimes \Phi_A)P = (P \otimes id_A)(id_A \otimes P)(\Phi)A \otimes id_B)$

Now, we will find a potential comultiplication $\Delta : A \otimes B \to A \otimes B \otimes A \otimes B$ by using a commutative diagram that represents multiplication. For the Zappa-Szep product of groups:



So, for quantum groups (A, Φ_A) and (B, Φ_B):



Hence, it would be reasonable to assume that $\Delta = (id_A \otimes P \otimes id_B)(\Phi_A \otimes \Phi_B)$.

5 Conjecture

Theorem 3 *Let (A, Φ_A) and (B, Φ_B) be compact quantum groups. Suppose $P : A \otimes B \to B \otimes A$ is an isomorphism satisfying:*

$$(\Phi_B \otimes id_A)P = (id_B \otimes P)(P \otimes id_B)(id_A \otimes \Phi_B)$$

$$(id_B \otimes \Phi_A)P = (P \otimes id_A)(id_A \otimes P)(\Phi)A \otimes id_B)$$

Let $\Delta : A \otimes B \to A \otimes B \otimes A \otimes B$ be given by

$$\Delta = (id_A \otimes P \otimes id_B)(\Phi_A \otimes \Phi_B)$$

Then $(A \otimes B, \Delta)$ is a compact quantum group.

Coassociativity

We want to show that $(id_{A\otimes B} \otimes \Delta)\Delta = (\Delta \otimes id_{A\otimes B})\Delta$

We know that

1. $(id_A \otimes \Phi_A)\Phi_A = (\Phi_A \otimes id_A)\Phi_A$
2. $(id_B \otimes \Phi_B)\Phi_B = (\Phi_B \otimes id_B)\Phi_B$
3. $(\Phi_B \otimes id_A)P = (id_B \otimes P)(P \otimes id_B)(id_A \otimes \Phi_B)$
4. $(id_B \otimes \Phi_A)P = (P \otimes id_A)(id_A \otimes P)(\Phi)A \otimes id_B)$

Now,

$(id_{A\otimes B} \otimes \Delta)\Delta = (id_A \otimes P \otimes id_{B\otimes A\otimes B})(id_{A\otimes A\otimes B} \otimes P \otimes id_B)(id_{A\otimes A} \otimes P \otimes id_{B\otimes B})(id_{A\otimes A\otimes A} \otimes \Phi_B \otimes id_B)(\Phi_A \otimes id_{A\otimes B\otimes B})(\Phi_A \otimes \Phi_B)$ [by 3]

$= (\mathrm{id}_A \otimes P \otimes id_{B\otimes A\otimes B})(id_{A\otimes A\otimes B} \otimes P \otimes id_B)(id_{A\otimes A} \otimes P \otimes id_{B\otimes B})(id_{A\otimes A\otimes A} \otimes \Phi_B \otimes id_B)(id_A \otimes \Phi_A \otimes id_{B\otimes B})(\Phi_A \otimes \Phi_B)$ [by 1]

$= (\mathrm{id}_A \otimes P \otimes id_{B\otimes A\otimes B})(id_{A\otimes A\otimes B} \otimes P \otimes id_B)(id_{A\otimes A} \otimes P \otimes id_{B\otimes B})(id_{A\otimes A\otimes A} \otimes id_B \otimes \Phi_B)(id_A \otimes \Phi_A \otimes id_{B\otimes B})(\Phi_A \otimes \Phi_B)$ [by 2]

$= (\mathrm{id}_A \otimes P \otimes id_{B\otimes A\otimes B})(id_{A\otimes A\otimes B} \otimes P \otimes id_B)(id_{A\otimes A} \otimes P \otimes id_{B\otimes B})(id_A \otimes \Phi_A \otimes id_{B\otimes B\otimes B})(id_{A\otimes A} \otimes id_B \otimes \Phi_B)(\Phi_A \otimes \Phi_B)$ [rearranging]

$= (\mathrm{id}_A \otimes id_B \otimes id_A \otimes P \otimes id_B)(id_A \otimes P \otimes id_A \otimes id_{B\otimes B})(id_{A\otimes A} \otimes P \otimes idB \otimes B)(id_A \otimes \Phi_A \otimes id_{B\otimes B\otimes B})(id_{A\otimes A} \otimes id_B \otimes \Phi_B)(\Phi_A \otimes \Phi_B)$ [rearranging]

$= (\mathrm{id}_A \otimes id_B \otimes id_A \otimes P \otimes id_B)(id_A \otimes [(P \otimes id_A)(id_A \otimes P)(\Phi_A \otimes id_B)] \otimes id_{B\otimes B})(id_{A\otimes A} \otimes id_B \otimes \Phi_B)(\Phi_A \otimes \Phi_B)$ [rearranging]

$= (\mathrm{id}_A \otimes id_B \otimes id_A \otimes P \otimes id_B)(id_A \otimes (id_B \otimes \Phi_A)P \otimes id_{B\otimes B})(id_{A\otimes A} \otimes id_B \otimes \Phi_B)(\Phi_A \otimes \Phi_B)$ [by 4]

$= (\mathrm{id}_A \otimes id_B \otimes id_A \otimes P \otimes id_B)(id_A \otimes (id_B \otimes \Phi_A) \otimes id_{B\otimes B})(id_A \otimes P \otimes id_{B\otimes B})(id_{A\otimes A} \otimes id_B \otimes \Phi_B)(\Phi_A \otimes \Phi_B)$ [rearranging]

$= (\mathrm{id}_A \otimes id_B \otimes id_A \otimes P \otimes id_B)(id_A \otimes (id_B \otimes \Phi_A) \otimes id_{B\otimes B})(id_A \otimes id_B \otimes id_A \otimes \Phi_B)(id_A \otimes P \otimes id_B)(\Phi_A \otimes \Phi_B)$ [rearranging]

$= (\mathrm{id}_A \otimes id_B \otimes id_A \otimes P \otimes id_B)(id_A \otimes id_B \otimes \Phi_A \otimes \Phi_B)(id_A \otimes P \otimes id_B)(\Phi_A \otimes \Phi_B)$ [rearranging]

$= (\Delta \otimes id_{A\otimes B})\Delta$

Density

We will prove that $(id_A \otimes P \otimes id_B)(\Phi_A \otimes \Phi_B)(A \otimes B)(1_A \otimes 1_B \otimes A \otimes B)$ is dense in $A \otimes B \otimes A \otimes B$.

Fix $\epsilon > 0$, and $\gamma_1 \otimes \tau_1 \otimes \gamma_2 \otimes \tau_2 \in A \otimes B \otimes A \otimes B$.

Choose $a_i \in A, w_i \in B$ satisfying

$||P^{-1}(\tau_1 \otimes \gamma_2) - \sum_{i=1}^m a_i \otimes w_i|| < \frac{\epsilon}{4||\gamma_1||||\tau_2||}$

Let $X_1 = \sum_{i=1}^m (id_A \otimes P \otimes id_B)(\gamma_1 \otimes a_i \otimes w_i \otimes \tau_2)$

We have $X_1 \in A \otimes B \otimes A \otimes B$.

Now,

$||\gamma_1 \otimes \tau_1 \otimes \gamma_2 \otimes \tau_2 - X_1|| = ||(id_A \otimes P \otimes id_B)(\gamma_1 \otimes P^{-1}(\tau_1 \otimes \gamma_2) \otimes \tau_2) - X_1||$

$= ||(id_A \otimes P \otimes id_B)(\gamma_1 \otimes (P^{-1}(\tau_1 \otimes \gamma_2) - \sum_{i=1}^m a_i \otimes w_i) \otimes \tau_2)||$

$= ||\gamma_1||||\tau_2||||P^{-1}(\tau_1 \otimes \gamma_2) - \sum_{i=1}^m a_i \otimes w_i||$

$< \frac{\epsilon}{4}$

For each $i \in 1, ..., m$, we choose $b_{ij}, c_{ij} \in A$ (using the fact that $\Phi_A(A)(1_A \otimes A)$ is dense in $A \otimes A$) such that

$||\gamma_1 \otimes a_i - \sum_{j=1}^{n} \Phi_A(b_{ij})(1_A \otimes c_{ij})|| < \frac{\epsilon}{4||\tau_2||m\max\{||w_i||:i\in\{1,\ldots,m\}\}}$

Let $X_2 = \sum_{i=1}^{m}\sum_{j=1}^{n}(id_A \otimes P \otimes id_B)(\Phi_A(b_{ij})(1_A \otimes c_{ij}) \otimes w_i \otimes \tau_2)$

We have

$||X_1 - X_2|| \leq \sum_{i=1}^{m} ||(id_A \otimes P \otimes id_B)(\gamma_1 \otimes a_i - \sum_{j=1}^{n} \Phi_A(b_{ij})(1_A \otimes c_{ij}) \otimes w_i \otimes \tau_2)||$

$= \sum_{i=1}^{n} ||w_i||||\tau_2||||\gamma_1 \otimes a_i - \sum_{j=1}^{n} \Phi_A(b_{ij})(1_A \otimes c_{ij})||$

$\leq m\max\{||w_i|| : 1 \leq i \leq m\}||\tau_2||||\gamma_1 \otimes a_i - \sum_{j=1}^{n} \Phi_A(b_{ij})(1_A \otimes c_{ij})||$

$< \frac{\epsilon}{4}$

Choose $d_{ijk} \in A, x_{ijk} \in B$ for $1 \leq i \leq m, 1 \leq j \leq m$ (using the assumption that $(1_A \otimes B)P^{-1}(1_B \otimes A)$ is dense in $B \otimes B$) such that

$||c_{ij} \otimes w_i - \sum_{k=1}^{p}(1_A \otimes x_{ijk})P^{-1}(1_B \otimes d_{ijk})|| < \frac{\epsilon}{4mn||\tau_2||\max\{||\Phi_A(b_{ij}:1\leq i\leq m,1\leq j\leq n\}}$

Let $X_3 = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p}((id_A \otimes P \otimes id_B)(\Phi_A(b_{ij}) \otimes x_{ijk} \otimes 1_B))(1_A \otimes 1_B \otimes d_{ijk} \otimes \tau_2)$

We have $X_3 = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p}(id_A \otimes P \otimes id_B)((\Phi_A(b_{ij})(1_A \otimes 1_B)(1_A \otimes [(1_A \otimes x_{ijk})P^{-1}(1_B \otimes d_{ijk})] \otimes \tau_2))$

So

$||X_2 - X_3|| \leq \sum_{i=1}^{m}\sum_{j=1}^{n} ||(id_A \otimes P \otimes id_B)(\Phi_A(b_{ij})(1_A \otimes c_{ij}) \otimes w_i \otimes \tau_2) - \sum_{k=1}^{p}((id_A \otimes P \otimes id_B)((\Phi_A(b_{ij} \otimes 1_B \otimes 1_B)(1_A \otimes ((1_A \otimes x_{ijk})P^{-1}(1_B \otimes d_{ijk})) \otimes \tau_2))||$

$= \sum_{i=1}^{m}\sum_{j=1}^{n} ||(\Phi_A(b_{ij}) \otimes 1_B \otimes 1_B)(1_A \otimes [c_{ij} \otimes w_i - \sum_{k=1}^{p}(1_A \otimes x_{ijk})P^{-1}(1_B \otimes d_{ijk})] \otimes \tau_2)||$

$= \sum_{i=1}^{m}\sum_{j=1}^{n} ||\Phi_A(b_{ij})||||\tau_2||||c_{ij} \otimes w_i - \sum_{k=1}^{p}(1_A \otimes x_{ijk})P^{-1}(1_B \otimes d_{ijk})|| < \frac{\epsilon}{4}$

For each $1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq p$ choose $y_{ijkl}, z_{ijkl} \in B$ (using the fact that $\Phi_B(B)(1_B \otimes B)$ is dense in $B \otimes B$) such that

$||x_{ijk} \otimes 1_B - \sum_{l=1}^{q} \Phi_B(y_{ijkl}(1_B \otimes z_{ijkl}))|| < \frac{\epsilon}{4mnp||\tau_2||\max\{||\Phi_A(b_{ij})||d_{ijk}||:1\leq i\leq m,1\leq j\leq n,1\leq k\leq p\}}$

Let $X_4 = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q}(id_A \otimes P \otimes id_B)(\Phi_A(b_{ij}) \otimes \Phi_B(y_{ijkl})(1_A \otimes z_{ijkl}))(1_A \otimes 1_B \otimes z_{ijkl} \otimes \tau_2)$

So

$||X_3 - X_4|| \leq \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} ||(id_A \otimes P \otimes id_B)(\Phi_A(b_{ij}) \otimes x_{ijk} \otimes 1_B)(1_A \otimes 1_B \otimes d_{ijk} \otimes \tau_2) - \sum_{l=1}^{q}(id_A \otimes P \otimes id_B)(\Phi_A(b_{ij}) \otimes \Phi_B(y_{ijkl})(1_B \otimes z_{ijkl}))(1_A \otimes 1_B \otimes d_{ijk} \otimes \tau_2)||$

$= \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} ||\Phi_A(b_{ij})|| ||x_{ijk} \otimes 1_B - \sum_{l=1}^{q} \Phi_B(y_{ijkl})(1_B \otimes z_{ijkl})|| ||d_{ijk}|| ||\tau_2||$

$< \frac{\epsilon}{4}$

Hence, $||\gamma_1 \otimes \tau_1 \otimes \gamma_2 \otimes \tau_2 - X_4|| = ||(\gamma_1 \otimes \tau_1 \otimes \gamma_2 \otimes \tau_2 - X_1) + (X_1 - X_2) + (X_2 - X_3) + (X_3 - X_4)||$

$< ||\gamma_1 \otimes \tau_1 \otimes \gamma_2 \otimes \tau_2 - X_1|| + ||X_1 - X_2|| + ||X_2 - X_3|| + ||X_3 - X_4||$

$< \epsilon$

References

[1] Maes, A., Van Daele, A.: *Notes on Compact Quantum Groups*, Nieuw Arch. Wisk. (4) 16 (1998), 73112.