Generalised Fractional Calculus

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Outline

1. Integration
2. Fractional Calculus
3. My Research

The Riemann sum

Partition of $[a, b]$: Non-overlapping intervals which cover $[a, b]$

$$a = x_0 < x_1 < \cdots < x_{i-1} < x_i < \cdots x_n = b$$

Tagged partition of $[a, b]$: Intervals in partition paired with tag points

$$P = \{([x_{i-1}, x_i], t_i)\}_{i=1}^{n}$$

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Riemann sum: P tagged partition, $f : [a, b] \to \mathbb{R}$

$$S(f; P) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$$

The Riemann integral

$\int_a^b f(x)dx = A$ means that:

For all $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that if a tagged partition P satisfies

$$0 < x_i - x_{i-1} < \delta_\varepsilon \quad \text{for} \quad i = 1, \ldots, n$$

then

$$|S(f, P) - A| < \varepsilon$$

The Generalised Riemann integral

$\int_a^b f(x)dx = A$ means that:

For all $\varepsilon > 0$, there exists a strictly positive function $\delta_\varepsilon : [a, b] \to \mathbb{R}_+$ such that if a tagged partition P satisfies

$$t_i \in [x_{i-1}, x_i] \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)] \quad \text{for} \quad i = 1, \ldots, n$$

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The Riemann-Stieltjes integral

Riemann-Stieltjes sum of f: Integrator $\alpha : I \to \mathbb{R}$, partition P

$$S(f, P, \alpha) = \sum_{i=1}^{n} f(t_i)[\alpha(x_i) - \alpha(x_{i-1})]$$

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The Generalised Riemann-Stieltjes integral

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Important points:

- δ is a function, not a constant
- Use of Riemann-Stieltjes sum

Motivation

Using this integral we can integrate a wide range of functions...

- $\alpha(x) = x \Rightarrow$ Generalised Riemann integral
- Riemann-Stieltjes $\Rightarrow$ Generalised Riemann-Stieltjes
- But not all Generalised Riemann-Stieltjes integrable functions are Riemann-Stieltjes integrable:

Example: $f, \alpha : [0,1] \to \mathbb{R}$

$$f(x) = \alpha(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ 1 & \text{if} \quad x \in (0,1] \end{cases}$$

Properties

- The collection of Generalised Riemann-Stieltjes integrable functions is a vector space
- Monotonicity
- Convergence Theorems - interchanging an integral with a limit. Analogues of MCT, DCT, Uniform CT, Mean CT, Fatou's Lemma, plus more!
- Hake's Theorem - improper integrals
- Simplification Theorem
- Fundamental Theorem of Calculus??

Existence Theorems

Which functions are integrable?

- Step functions
- Regulated functions
- Continuous functions and monotone functions

But...we need α to be Lipschitz continuous!

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Products of functions?

- f bounded below
- g regulated
- α Lipschitz and increasing

$\Rightarrow fg$ integrable!

The α-derivative

Why haven't we seen the Fundamental Theorem of Calculus yet??

Since we are integrating with respect to α, it also makes sense to consider derivatives with respect to α...

Definition

α continuous and strictly increasing
f is α-differentiable at x_0 if

$$D_\alpha f(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{\alpha(x) - \alpha(x_0)}$$

exists.

Properties

All the usual results still hold...

- Algebra of differentiable functions
- Chain rule, product rule, quotient rule
- Mean Value Theorems
- Characterisation of maxima/minima

How does it relate to the ordinary derivative?

Theorem

$$f, \alpha \text{ } nice \Rightarrow D_\alpha f(x_0) = \frac{f'(x_0)}{\alpha'(x_0)}$$

The Fundamental Theorem

Theorem

- α *continuous* and *strictly increasing*
- *F continuous*
- $D_\alpha F(x) = f(x)$ *for all* $x \in I$

Then f integrable and $\int_a^b f d\alpha = F(b) - F(a)$

Why is this important?

- Integrability as consequence, not hypothesis
- Integration by parts
- Substitution theorems
- New result!

Fractional Calculus

Motivation: Familiar with $D_x f, D_x^2 f, D_x^3 f, ... D_x^k f$.
What about $D_x^\mu f$ for $\mu \in \mathbb{R}$?
Many ways to approach this problem...

1) (Lacroix, 1819) For $f(x) = x^n$, we have

$$\frac{d^k f(x)}{dx^k} = \frac{n!}{(n-k)!} x^{n-k}$$

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Factorials $\rightarrow$ Gamma function

$$\frac{d^k f(x)}{dx^k} = \frac{\Gamma(n+1)}{\Gamma(n-k+1)} x^{n-k}$$

2) (Liouville, 1832) We have

$$D_x^k e^{ax} = a^k e^{ax}$$

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For functions

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3) (Euler, 1730) Interpolation between integer derivatives

Today's definitions

Fractional integral:
Recall Dirichlet's Theorem for n-fold integration:

$$\frac{1}{(n-1)!}\int_0^x (x-u)^{n-1} f(u)du$$

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Replace factorial with Γ, replace k with μ

$$D^{-\mu} f(x) = \frac{1}{\Gamma(\mu)}\int_0^x (x-u)^{\mu-1} f(u)du$$

(provided it exists)

Fractional derivatives

Riemann-Liouville:
(integrate then differentiate)

$$D^{\mu} f(x) = D^{\lceil \mu \rceil}[D^{-(\lceil \mu \rceil - \mu)} f(x)]$$

Caputo:
(differentiate then integrate)

$$D^{\mu} f(x) = D^{-(\lceil \mu \rceil - \mu)}[D^{\lceil \mu \rceil} f(x)]$$

Generalised Fractional Calculus

What is it?

- Riemann integral $\rightarrow$ Generalised Riemann-Stieltjes integral
- Ordinary derivative $\rightarrow$ α-derivative

Motivation:

- Easier to prove existence results
- Easier to deal with improper integrals (Hake's Theorem!)
- Can work with a wider range of functions
- ...and more! (Future work)

Fractional α integral

$$D_\alpha^{-\mu} f(x) = \frac{1}{\Gamma(\mu)} \int_0^x (x-u)^{\mu-1} f(u) d\alpha(u)$$

Theorem

α Lipschitz and increasing, f regulated or bounded below $\Rightarrow$ integral exists.

Fractional α derivatives

Riemann-Liouville derivative wrt α

$$RL(f, \alpha, \mu) = D_\alpha^{\lceil \mu \rceil} D_\alpha^{-(\lceil \mu \rceil - \mu)} f(x)$$

Existence? Problem - α-derivative. Very specific α needed.

Fractional α derivatives

Riemann-Liouville derivative wrt α

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Caputo derivative wrt α

$$\begin{aligned} C(f, \alpha, \mu)(x) &= D_\alpha^{-(\lceil \mu \rceil - \mu)} D_\alpha^{\lceil \mu \rceil} f(x) \\ &= \int_0^x (x-u)^{-(\lceil \mu \rceil - \mu)} [D_\alpha^{\lceil \mu \rceil} f(u)] d\alpha(u) \end{aligned}$$

Theorem

α Lipschitz and increasing, f "C^n" $\Rightarrow$ integral exists.

Example: Functions that ordinary fractional calculus can't handle!

Functions $\alpha, f : [0,1] \to \mathbb{R}$:

$$\alpha(x) = \begin{cases} x & \text{if} \quad 0 \leq x \leq 0.5 \\ x^2 + 0.5 & \text{if} \quad 0.5 < x \leq 1 \end{cases}$$

$$f(x) = \begin{cases} 0 & \text{if} \quad 0 \leq x \leq 0.25 \\ 1 & \text{if} \quad 0.25 < x \leq 1 \end{cases}$$

- Can't use the Simplification Theorem
- But the fractional α-integral exists!

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Next steps?

- Existence
- Geometric interpretation
- Fractional differential equations

References

- Bartle, R. (2001). A modern theory of integration. Providence (R.I.): American mathematical Society.
- Bartle, R. and Sherbert, D. (2011). Introduction to real analysis. [New York]: J. Wiley and Sons.
- Widder, D. (1989). Advanced calculus. New York: Dover Publ.
- Apostol, T. (1974). Mathematical analysis. 2nd ed. Reading, Mass., etc.: Addison-Wesley.
- Bartle, R. (1996). Return to the Riemann Integral. The American Mathematical Monthly, 103(8), p.625.
- Castillo, D., Chapinz, S. (2008). The Fundamental Theorem of Calculus for the Riemann-Stieltjes Integral. Lecturas Matematicas, p115.

Lebesgue integrability

There are functions which are not Riemann or Lebesgue integrable, but are Generalised Riemann integrable.

Example: The function $F : [0, 1] \to \mathbb{R}$ defined by

$$F(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ x^2 cos(\frac{\pi}{x^2}) & \text{if} \quad x \in (0, 1] \end{cases}$$

is differentiable on [0,1]. The Fundamental Theorem for Generalised Riemann integrable functions implies that its derivative is Generalised Riemann integrable. However, F is not absolutely continuous, so f is not Lebesgue integrable.

Hake's Theorem

Theorem

A function $f : I \to \mathbb{R}$ is integrable if and only if there exists $A \in \mathbb{R}$ such that for all $c \in (a, b)$ the restriction of f to $[a, c]$ is integrable and

$$\lim_{c \to b^-} \int_a^c f d\alpha = A$$

We then have

$$A = \int_a^b f d\alpha$$

The Simplification Theorem

Theorem

Let $\alpha : [a, b] \to \mathbb{R}$ *be an increasing function with* $\alpha \in C^1([a, b])$*.*
Let $f : [a, b] \to \mathbb{R}$ *be a continuous function with* $f \in GRS([a, b], \alpha)$*. Then*

$$\int_a^b f(x)d\alpha(x) = \int_a^b f(x)\alpha'(x)dx$$

where the integral on the left is a Generalised Riemann-Stieltjes integral, and the integral on the right is a Generalised Riemann integral.

Convergence

(Monotone Convergence Theorem)
Let $\alpha : I \to \mathbb{R}$ be an increasing function. Let $(f_k)_{k=1}^{\infty}$ be a monotone sequence in $GRS(I, \alpha)$. Let $f(x) := \lim_{k\to\infty} f_k(x)$ for all $x \in I$. Then $f \in GRS(I, \alpha)$ if and only if the sequence $(\int_I f_k d\alpha)_{k=1}^{\infty}$ is bounded in $\mathbb{R}$. Furthermore,

$$\int_I f d\alpha = \lim_{k\to\infty} \int_I f_k d\alpha$$

(Dominated Convergence Theorem)
Let $\alpha : I \to \mathbb{R}$ be an increasing function. Let (f_k) be a sequence in $GRS(I, \alpha)$ with $f(x) := \lim f_k(x)$ for all $x \in I$. Suppose that there exist functions $\beta, w \in GRS(I, \alpha)$ such that $\beta(x) \leq f_k(x) \leq w(x)$ for all $x \in I, k \in \mathbb{N}$. Then $f \in GRS(I, \alpha)$ and

$$\int_I f d\alpha = \lim_{k\to\infty} \int_I f_k d\alpha$$