Fractional Calculus and Generalisations of the Riemann Integral

Penelope Drastik

Supervisor Marianito Rodrigo

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Chapter 1

Introduction

From the very first work on integration involving the Riemann integral, to today's standard Lebesgue integral, there have been many different definitions proposed for constructing a useful and practical integral. While Riemann integration is easy to understand, it has many limitations in both the functions which can be integrated and the ways in which the integral can be used (for example, its interaction with limits). On the other hand, Lebesgue integration is powerful, but as argued by Bartle in [5], requires a lot of background to define, making it difficult to think about intuitively and use practically. His proposed solution was the Generalised Riemann integral, a construction first invented in the 1950s by Kurzweil and Henstock. This integral could handle a wider range of functions than either the Riemann or Lebesgue integrals, was easy to define, reasonably easy to compute, and many theorems concerning the Riemann and Lebesgue integrals had analogues in the Generalised Riemann context.

The Riemann-Stieltjes integral is a generalisation of the Riemann integral in that it involves two functions - both an integrand and an integrator. In this thesis, we will be focussing on the Generalised Riemann-Stieltjes integral, a little-known type of integral which aims to do for the Riemann-Stieltjes integral what Kurzweil, Henstock and Bartle did for the Riemann integral.

The aim of our investigation of this integral is to apply it to the field of fractional calculus. As early as the 1600s, mathematicians were questioning the possibility of taking derivatives of a fractional order. Today, fractional calculus has expanded to include both real and complex order derivatives. Over the years, many definitions of a “fractional” derivative have been proposed. These methods had advantages and disadvantages compared to today’s ideas - the main positive being ease of calculuation, and the main negative being the limited range of functions to which the definitions could be applied. We will examine today’s definitions, and then proceed to apply the Generalised Riemann-Stieltjes integral and a corresponding derivative to generalise those definitions. The goal of this work is to provide new existence theorems, apply the ideas of fractional calculus to functions which have not been used in standard practice, and create a framework for fractional calculus on infinite intervals.

In Chapter 2, we review the definition of the Riemann integral, and then examine three generalisations of it: the Generalised Riemann Integral (also known as the Kurzweil-Henstock integral), the Riemann-Stieltjes integral and the Generalised Riemann-Stieltjes integral (also known as the Henstock-Stieltjes integral, such as in [8]). We will also see some typical examples of various types of integration.

In Chapter 3, we study the Generalised Riemann-Stieltjes integral in closer detail, proving a number of useful properties and presenting theorems concerning Riemann or Lebesgue integration based on the proofs given in [1] and [2]. We also introduce interval gauges based on the corresponding concept for the Generalised Riemann integral in [2].

In Chapter 4, we present the α-derivative, a generalisation of the ordinary derivative which is intended for use with Riemann-Stieltjes or Generalised Riemann-Stieltjes integrals as the natural "inverse" of the α-integral. We follow and expand upon the results in [6], with original work as well as several theorems based on results concerning the standard derivative. We also see some examples in order to better understand this little-known type of derivative.

In Chapter 5, we prove the Fundamental Theorem of Calculus for the Generalised Riemann-Stieltjes integral, an original result based upon the proofs in [2] and [6] for the Generalised Riemann and Riemann-Stieltjes integrals respectively.

In Chapter 6, we examine a number of convergence results including theorems corresponding to the familiar Monotone and Dominated Convergence Theorems following [2].

In Chapter 7, we prove an original result which compares Generalised Riemann-Stieltjes integrals to Riemann or Generalised Riemann integrals in an attempt to simplify the calculations involved based on a similar theorem concerning the Riemann-Stieltjes integral in [3] and [4]. The modifications in our proof are based on the techniques presented in [2].

In Chapter 8, we prove a number of existence results for the Generalised Riemann-Stieltjes integral based on similar methods in [2].

In Chapter 9, we examine the concept of improper integrability via Hake's Theorem based on [1]. As a prerequisite, we review the Vitali Covering Theorem. We briefly examine Generalised Riemann-Stieltjes integration on infinite intervals.

In Chapter 10, we prove a property of the Generalised Riemann-Stieltjes integral which converts between integration and summation in appropriate circumstances based on a similar property for the Riemann-Stieltjes integral in [3] and [4]. In an original application of this result, we rewrite the definitions of the Laplace and Z-transforms. We also apply the result in some original work which provides a discrete approximation of an integral used in fractional calculus.

In Chapter 11, we review fractional calculus and provide some examples of typical techniques as in [9].

In Chapter 12, we introduce generalised fractional calculus via revised definitions of the Riemann-Liouville fractional integral and derivative, and the Caputo fractional derivative. The work in this chapter is completely original. It is motivated by the question of whether we can generalise concepts from fractional calculus using the α-derivative and Generalised Riemann-Stieltjes integral. We prove appropriate existence results based on the work in Chapter 8, and give some examples.

Chapter 2

A Variety of Integrals

The Riemann Integral

To begin, we shall review the definition of the familiar Riemann integral since the definitions of the three other integrals we will consider are based upon it.

Let $I = [a, b]$.

Definition 1. A **partition** of I is a finite collection $\{I_i\}_{i=1}^n$ of non-degenerate closed intervals $I_i = [x_{i-1}, x_i]$ satisfying

$$a = x_0 < x_1 < \cdots < x_{i-1} < x_i < \cdots < x_n = b$$

Example 1. Let $I = [0, 1]$. The collection $\{I_i\}_{i=1}^n$, where $I_i = [\frac{i-1}{n}, \frac{i}{n}]$ is a partition of I.

Definition 2. A **tagged partition** P of I is a finite collection of ordered pairs $P = \{(I_i, t_i)\}_{i=1}^n$ where $\{I_i\}_{i=1}^n$ is a partition of I and $t_i \in I_i$ for all $i \in \{1, \cdots, n\}$. The t_i are called **tags**.

Example 2. Using the same partition of $[0, 1]$ from Example 1, we assign tags $t_i = \frac{i}{n}$. Then $P = \{(I_i, t_i)\}_{i=1}^n$ is a tagged partition of $[0, 1]$. Note that in this example we assigned each interval its right endpoint as a tag, however this choice was completely arbitrary. We could easily have chosen the left endpoint, midpoint or any other point in the interval. We could also choose the left endpoint for some intervals, and the right endpoint for others.

Definition 3. If $P = \{(I_i, t_i)\}_{i=1}^n$ is a tagged partition of I and $f : I \to \mathbb{R}$ is a function, then the **Riemann sum** of f with respect to P is given by

$$S(f; P) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$$

Example 3. We will use the tagged partition of $[0, 1]$ given in Example 2, and choose the function $f : [0, 1] \to \mathbb{R}$ given by $f(x) = x$. The Riemann sum of f with respect to P is given by

$$\begin{aligned} S(f, P) &= \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{i}{n} - \frac{i-1}{n}\right) \\ &= \sum_{i=1}^{n} \frac{i}{n}\left(\frac{1}{n}\right) \\ &= \frac{1}{n^2} \sum_{i=1}^{n} i \\ &= \frac{1}{n^2}\left(\frac{1}{2}n(n+1)\right) \\ &= \frac{n+1}{2n} \end{aligned}$$

Definition 4. The function $f : I \to \mathbb{R}$ is **Riemann integrable** if there exists $L \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists a constant $\delta_\varepsilon > 0$ such that if P is a tagged partition of I satisfying

$$0 < x_i - x_{i-1} < \delta_\varepsilon$$

for $i = 1, \ldots, n$, then

$$|S(f, P) - L| < \varepsilon$$

We denote the space of Riemann integrable functions on $[a,b]$ by $R[a,b]$, and write $L = \int_a^b f$.

Example 4. Consider the function $f : [a,b] \to \mathbb{R}$ given by $f(x) = c \quad \forall x \in [a,b]$, where c is a constant. We will show that $f \in R[a,b]$:
Fix $\varepsilon > 0$. Let us choose $\delta_\varepsilon := 1$. Suppose that $P = \{(I_i, t_i)\}$ is a tagged partition of $[a,b]$ satisfying $0 < x_i - x_{i-1} < \delta_\varepsilon$ for $i \in \{1, \ldots, n\}$. We calculate the Riemann sum

$$S(f,P) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} c(x_i - x_{i-1}) = c(b-a)$$

So we have $|S(f,P) - c(b-a)| = 0 < \varepsilon$, and since ε was arbitrary, we conclude that $f \in R[a,b]$ and $\int_a^b f = c(b-a)$.

Example 5. We return to Example 3. We will show that $f \in R[0,1]$ and $\int_a^b f = \frac{1}{2}$.
Fix $\varepsilon > 0$. Let us choose $\delta_\varepsilon := \varepsilon$. Suppose that $P = \{(I_i, t_i)\}_{i=1}^n$ is a tagged partition of $[0,1]$ which satisfies $x_i - x_{i-1} < \delta_\varepsilon$ for all $i \in \{1, \ldots, n\}$.
Now let us define another tagged partition, $Q = \{(I_i, q_i)\}_{i=1}^n$, where $q_i = \frac{1}{2}(x_i + x_{i-1})$. So q_i is the midpoint of the interval I_i.
We have

$$\begin{aligned} S(f,Q) &= \sum_{i=1}^{n} f(q_i)(x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} q_i(x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} \frac{1}{2}(x_i + x_{i-1})(x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} \frac{1}{2}(x_i^2 - x_{i-1}^2) \\ &= \frac{1}{2}(1^2 - 0^2) \\ &= \frac{1}{2} \end{aligned}$$

since the last summation is a telescoping sum.

Now, note that $|t_i - q_i| < \delta_\varepsilon$ for all $i \in \{1, \ldots, n\}$ since $t_i, q_i \in I_i$ and the length of each of the I_i is bounded by δ_ε.

We now use the triangle inequality to relate $S(f, P)$ to $S(f, Q)$:

$$\begin{aligned}
|S(f,P) - S(f,Q)| &= \left| \sum_{i=1}^{n} t_i(x_i - x_{i-1}) - \sum_{i=1}^{n} q_i(x_i - x_{i-1}) \right| \\
&\leq \sum_{i=1}^{n} |t_i - q_i|(x_i - x_{i-1}) \\
&< \delta_\varepsilon \sum_{i=1}^{n} (x_i - x_{i-1}) \\
&= \delta_\varepsilon (x_n - x_0) \\
&= \delta_\varepsilon
\end{aligned}$$

So we have $|S(f,P) - S(f,Q)| = |S(f,P) - \frac{1}{2}| < \delta_\varepsilon = \varepsilon$, and therefore $f \in R[0,1]$ and $\int_a^b f = \frac{1}{2}$.

The Generalised Riemann Integral

The next integral we shall see is the Generalised Riemann integral, otherwise known as the Henstock-Stieltjes integral, which was invented in the 1950's and popularised in the 1990's-2000's in a series of articles and textbooks by Robert Bartle. In his work, Bartle argued that this integral should be more widely used since it encompasses both Riemann and Lebesgue integrable functions, as well as other functions which neither of those integrals can handle. In a result known as the Consistency Theorem, it is shown that all Riemann integrable functions are Generalised Riemann integrable, and the integration evaluates to the same result. It is also known that a function is Lebesgue integrable if and only if both f and $|f|$ are Generalised Riemann integrable. This integral also deals with improper integrals better, as the limiting process we're familiar with regarding the Riemann integral does not extend the Generalised Riemann integral, meaning that integrability and improper integrability essentially amount to the same thing. We will not go into properties of this integral in detail as many of them are reflected in theorems about the Generalised Riemann-Stieltjes integral that we will see later.

Definition 5. A **gauge** on I is a strictly positive function $\delta : I \to (0, \infty)$.

Example 6. The function $\delta : [0, 1] \to (0, \infty)$ given by

$$\delta(x) = \begin{cases} 1 & \text{if} \quad x = 0 \\ x & \text{if} \quad x \in (0, 1] \end{cases}$$

is a gauge.

Definition 6. If $P = \{(I_i, t_i)\}_{i=1}^{n}$ is a tagged partition of I, and $\delta : I \to (0, \infty)$ is a gauge on I, then P is **δ-fine** if

$$t_i \in I_i \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)]$$

for all $i \in \{1, \ldots, n\}$.

Example 7. Consider the gauge δ from example 6. The partition $P = \{([0, 0.75], 0), ([0.75, 1], 0.8)\}$ is δ-fine since

$$x_1 - x_0 = 0.75 < \delta(t_0) = \delta(0) = 1$$

and

$$x_2 - x_1 = 1 - 0.75 = 0.25 < \delta(t_1) = \delta(0.8) = 0.8$$

Definition 7. The function $f : I \to \mathbb{R}$ is **Generalised Riemann integrable** if there exists $L \in \mathbb{R}$ such that if for every $\varepsilon > 0$ there exists a gauge $\delta_\varepsilon : I \to (0, \infty)$ such that if P is a tagged partition of I which is δ_ε-fine, then

$$|S(f, P) - L| < \varepsilon$$

We denote the space of Generalised Riemann integrable functions on I by $GR(I)$, and write $L = \int_a^b f$.

The Riemann-Stieltjes Integral

This integral is a generalisation of the Riemann integral because we are now working with two functions (an integrand and an integrator) rather than just one.

Firstly, we need to modify the concept of the Riemann sum to include the integrator.

Definition 8. The **Riemann-Stieltjes sum** of a function $f : I \to \mathbb{R}$ with respect to the integrator function $\alpha : I \to \mathbb{R}$ and the partition $P = \{(I_i, t_i)\}_{i=1}^n$ is given by

$$S(f, P, \alpha) = \sum_{i=1}^{n} f(t_i)[\alpha(x_i) - \alpha(x_{i-1})]$$

Definition 9. The function $f : I \to \mathbb{R}$ is **Riemann-Stieltjes integrable** with respect to the integrator function $\alpha : I \to \mathbb{R}$ if there exists $L \in \mathbb{R}$ such that for all $\varepsilon > 0$ there exists a constant $\delta_\varepsilon > 0$ such that if $P = \{(I_i, t_i)\}_{i=1}^n$ is a tagged partition of I satisfying

$$0 < x_i - x_{i-1} < \delta_\varepsilon$$

for all $i \in \{1, \dots n\}$, then

$$|S(f, P, \alpha) - L| < \varepsilon$$

We denote the space of functions on I which are Riemann-Stieltjes integrable with respect to α by $RS(I, \alpha)$, and write $L = \int_a^b f d\alpha$.

Note that while the definition does not place any restrictions on the integrator function α, we naturally need some extra conditions on both α and f for the integral to exist. We have the following result: If α is monotone and f is continuous then $\int_a^b f d\alpha$ exists. However, there are a number of useful integrator functions outside of the context of this theorem, such as step functions, which we will see much later using the Generalised Riemann-Stieltjes integral.

The Generalised Riemann-Stieltjes Integral

The final integral we will see combines elements of the definitions of all of the previous integrals we have examined, in the hopes that we will also be able to combine useful properties of those integrals.

Definition 10. The function $f : I \to \mathbb{R}$ is **Generalised Riemann-Stieltjes integrable** with respect to integrator function $\alpha : I \to \mathbb{R}$ if there exists $L \in \mathbb{R}$ such that for all $\varepsilon > 0$ there exists a gauge $\delta_\varepsilon : I \to (0, \infty)$ such that if $P = \{(I_i, t_i)\}_{i=1}^n$ is a tagged partition of I which is δ_ε-fine, then

$$|S(f, P, \alpha) - L| < \varepsilon$$

We denote the space of functions which are Generalised Riemann-Stieltjes integrable on I with respect to α by $GRS(I)$, and write $L = \int_a^b f d\alpha$.

Remark. When the integrator function is given by $\alpha(x) = x$, the Generalised Riemann-Stieltjes integral reduces to the Generalised Riemann integral.

Example 8. The Generalised Riemann-Stieltjes integral is useful because it allows us to integrate a wider range of functions than we could with the Riemann-Stieltjes integral. Here is an example of a function which is not Riemann-Stieltjes integrable with respect to an integrator function α, but is Generalised Riemann-Stieltjes integrable with respect to α.
Let $[a, b] = [0, 1]$. We define functions $f, \alpha : [0, 1] \to \mathbb{R}$ by

$$f(x) = \alpha(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \in (0, 1] \end{cases}$$

Claim: The function f is not Riemann-Stieltjes integrable with respect to α.
Proof of Claim: Let $P = \{(I_i, t_i)\}_{i=1}^n$ be a tagged partition of $[0, 1]$. For the first interval $[x_0, x_1]$ we have

$$\alpha(x_1) - \alpha(x_0) = \alpha(x_1) - \alpha(0) = 1 - 0 = 1$$

and for all other intervals, we have

$$\alpha(x_i) - \alpha(x_{i-1}) = 1 - 1 = 0$$

Now, the Riemann-Stieltjes sum can have two values, depending on how the tag for the first interval, t_1, is chosen. If $t_1 = 0$ then $f(t_1) = 0$. If $t_1 > 0$ then $f(t_1) = 1$. So the Riemann-Stieltjes sum is given by

$$\begin{aligned} S(f, P, \alpha) &= \sum_{i=1}^{n} f(t_i)[\alpha(x_i) - \alpha(x_{i-1}] \\ &= \begin{cases} 0 & \text{if} \quad t_1 = 0 \\ 1 & \text{if} \quad t_1 \in (0, x_1] \end{cases} \end{aligned}$$

Since the sum has two different values, the absolute value of the difference between $S(f, P, \alpha)$ and any $A \in \mathbb{R}$ cannot be less than any ε, and therefore f is not Riemann-Stieltjes integrable with respect to α.
Claim: The function f is Generalised Riemann-Stieltjes integrable with respect to α.
Proof of Claim: Fix $\varepsilon > 0$. We choose the gauge

$$\delta(x) = \begin{cases} \frac{1}{4} & \text{if} \quad x = 0 \\ \frac{x}{2} & \text{if} \quad x \in (0, 1] \end{cases}$$

Now, suppose that $P = \{(I_i, t_i)\}$ is a δ-fine partition of I. We claim that the tag for the first interval in the partition, that is, $[0, x_1]$ for some $x_1 \in I$, must be $t_1 = 0$. Since P is δ-fine, we have

$$[0, x_1] \subseteq [t_1 - \delta(t_1), t_1 + \delta(t_1)]$$

Hence $t_1 - \delta(t_1) \leq 0$. Now, if $t_1 > 0$ then $\delta(t_1) = \frac{t_1}{2}$ and so

$$t_1 - \delta(t_1) = t_1 - \frac{t_1}{2} = \frac{t_1}{2} > 0$$

This creates a contradiction, so we must have $t_1 = 0$. The Riemann-Stieltjes sum is given by

$$S(f, P, \alpha) = f(t_1)[\alpha(x_1) - \alpha(0)] = 0$$

So the Riemann-Stieltjes sum approaches a limit, and therefore the Generalised Riemann-Stieltjes integral exists.

Theorem 1. *Let $\alpha : I \to \mathbb{R}$ be any integrator function, and let $f : I \to \mathbb{R}$ be given by $f(x) = c$ for all $x \in I$, where $c \in \mathbb{R}$ is a constant. Then $f \in GRS(I, \alpha)$ and*

$$\int_a^b f d\alpha = c[\alpha(b) - \alpha(a)]$$

Proof. Fix $\varepsilon > 0$. We define the gauge $\delta_\varepsilon : I \to (0, \infty)$ by $\delta_\varepsilon(t) := 1$ for all $t \in I$.

We have

$$\begin{aligned} S(f, P, \alpha) &= \sum_{i=1}^{n} f(t_i)[\alpha(x_i) - \alpha(x_{i-1})] \\ &= \sum_{i=1}^{n} c[\alpha(x_i) - \alpha(x_{i-1})] \\ &= c \sum_{i=1}^{n} [\alpha(x_i) - \alpha(x_{i-1})] \\ &= c[\alpha(b) - \alpha(a)] \end{aligned}$$

So we have $|S(f, P, \alpha) - c[\alpha(b) - \alpha(a)]| = 0 < \varepsilon$, and therefore $f \in GRS(I, \alpha)$ and

$$\int_a^b f d\alpha = c[\alpha(b) - \alpha(a)]$$

□

Chapter 3

Properties of the GRS Integral

In this section, we will prove a number of basic properties of the Generalised Riemann-Stieltjes integral. Although they are important in themselves, a number of these results are essential building blocks in other theorems, as well as aids in the computation of integrals, something which is generally difficult to do just using the definition.

We first see a result which is important in the theory of both the Generalised Riemann-Stieltjes and Generalised Riemann integrals.

Theorem 2. *(Cousin's Theorem) If* $\delta : I \to (0, \infty)$ *is a gauge on* I *then there exists a* δ*-fine partition of* I*.*

Proof. We define

$$E := \{x \in [a, b] : \quad \text{there exists a} \quad \delta\text{-fine partition of} \quad [a, x]\}$$

Now E is non-empty since $([a, x], a)$ is a δ-fine partition of $[a, x]$ for $x \in [a, a + \delta(a)] \cap [a, b]$. Since $E \subseteq [a, b]$, the set E is bounded. Let $u = \sup(E)$, so that $a < u \leq b$.

Claim: $u \in E$.

Proof of Claim: Since $u - \delta(u) < u = \sup(E)$, there exists $v \in E$ such that

$$u - \delta(u) < v < u$$

Let P_1 be a δ-fine partition of $[a, v]$. Define $P_2 := P_1 \cup ([v, u], u)$. Then P_2 is a δ-fine partition of $[a, u]$, since P_1 is δ-fine and we have

$$u - \delta(u) < v < u \leq u < u + \delta(u)$$

Hence $u \in E$.

Claim: $u = b$.

Proof of Claim: Suppose $u < b$. Let $w \in [a, b]$ be such that

$$u < w < u + \delta(u)$$

If Q_1 is a δ-fine partition of $[a, u]$, then define $Q_2 := Q_1 \cup ([u, w], u)$. Now Q_2 is a δ-fine partition of $[a, w]$ since Q_1 is δ-fine and

$$u - \delta(u) < u \leq u < w < u + \delta(u)$$

So we have $w \in E$. But this contradicts the assumption that u is an upper bound of E, therefore we conclude that $u = b$. □

We now see another way to define the concept of a gauge, which can be applied to both the Generalised Riemann and Generalised Riemann-Stieltjes integrals.

Definition 11. An **interval gauge** on an interval I is a mapping $\Delta : t \to \Delta(t) = [a(t), b(t)]$ such that $t \in (a(t), b(t))$ for all $t \in I$.

Example 9. The mapping on $[0, 1]$ given by $\Delta : t \to [t - 1, t + 1]$ is an interval gauge.

Remark. If Δ is an interval gauge on I then the mapping

$$\delta_\Delta : I \to [0, \infty)$$

defined by $\delta_\Delta(t) := \min\{t - a(t), b(t) - t\}$ for $t \in I$ is a (point) gauge.

Definition 12. Suppose that Δ is an interval gauge on I. A tagged partition $P = \{(I_i, t_i)\}_{i=1}^n$ is Δ-fine if

$$I_i \subseteq \Delta(t_i)$$

for all $i = 1, \ldots, n$.

Theorem 3. *Let I be an interval and let Δ be an interval gauge on I. Then there exists a Δ-fine tagged partition of I.*

Proof. Consider the gauge δ_Δ defined in Remark 3. By Cousin's Theorem, there exists a δ_Δ-fine tagged partition P of I. Now, if a partition is δ_Δ-fine then it is also Δ-fine, and so P is a Δ-fine tagged partition. □

Definition 13. A function $f : I \to \mathbb{R}$ is **Generalised Riemann-Stieltjes interval integrable** with respect to a function $\alpha : I \to \mathbb{R}$ on I to a value of $D \in \mathbb{R}$ if for all $\varepsilon > 0$ there exists an interval gauge Δ_ε on I such that if a tagged partition P is Δ_ε-fine then

$$|S(f, P, \alpha) - D| < \varepsilon$$

Theorem 4. *We have $f \in GRS(I, \alpha)$ with integral D if and only if f is Generalised Riemann-Stieltjes interval integrable on I with value D.*

Proof. Suppose that $f \in GRS(I, \alpha)$ with integral D. We can convert the point gauge δ_ε into an interval gauge Δ_ε by setting $\Delta_\varepsilon(t) := [t - \delta_\varepsilon(t), t + \delta_\varepsilon(t)]$ to show that f is Generalised Riemann-Stieltjes integrable with value D. Suppose f is Generalised Riemann-Stieltjes interval integrable on I with value D. Then, using the point gauge in Remark 3 we conclude that $f \in GRS(I, \alpha)$ with value D. □

The next result is important because it relates Generalised Riemann-Stieltjes integrability with Riemann-Stieltjes integrability, much in the same way that we can relate Generalised Riemann and Riemann integrability. It also provides a helpful aid for calculation.

Theorem 5. *(Consistency Theorem) If $f \in RS([a,b],\alpha)$ with $\int_a^b f d\alpha = L$ then $f \in GRS([a,b],\alpha)$ with $\int_a^b f d\alpha = L$.*

Proof. Fix $\varepsilon > 0$. We need to find a suitable gauge. Since $f \in RS([a,b],\alpha)$, there exists a constant $\delta_\varepsilon > 0$ such that if $P = \{(I_i, t_i)\}_{i=1}^n$ is a tagged partition with

$$0 < x_i - x_{i-1} < \delta_\varepsilon$$

for all $i \in \{1, \dots, n\}$ then $|S(f, P, \alpha) - L| < \varepsilon$. We define $\delta_\varepsilon^*(t) := \frac{1}{4}\delta_\varepsilon$ for all $t \in [a,b]$. If P^* is a δ_ε^*-fine partition then

$$I_i \subseteq [t_i - \delta_\varepsilon^*(t_i), t_i + \delta_\varepsilon^*(t_i)] = [t_i - \frac{\delta_\varepsilon}{4}, t_i + \frac{\delta_\varepsilon}{4}]$$

This is equivalent to

$$t_i - \frac{\delta_\varepsilon}{4} \leq x_{i-1} \leq t_i \leq x_i \leq t_i + \frac{\delta_\varepsilon}{4}$$

So we have

$$0 < x_i - x_{i-1} \leq (t_i + \frac{\delta_\varepsilon}{4}) - (t_i - \frac{\delta_\varepsilon}{4}) = \frac{\delta_\varepsilon}{2} < \delta_\varepsilon$$

for all $i \in \{1, \dots, n\}$. Hence P^* satisfies the condition from the definition of the Riemann-Stieltjes integral, and so

$$|S(f, P^*, \alpha) - L| < \varepsilon$$

So every δ_ε-fine partition satisfies $|S(f, P^*, \alpha) - L| < \varepsilon$, and since $\varepsilon > 0$ is arbitrary, we conclude that $f \in GRS(I)$ and $\int_a^b f d\alpha = L$. □

Now we shall prove that the result of a Generalised Riemann-Stieltjes integration is unique, an important theorem without which this integral would not make sense.

Theorem 6. *(Uniqueness Theorem) If $f \in GRS([a,b],\alpha)$ then the value of the integral $\int_a^b f d\alpha$ is uniquely determined.*

Proof. Assume that L_1, L_2 both satisfy the definition. Fix $\varepsilon > 0$. Since L_1 satisfies the definition, there exists $\delta^1_{\varepsilon/2} : [a,b] \to (0,\infty)$ such that if P_1 is a $\delta^1_{\varepsilon/2}$-fine partition then

$$|S(f, P_1, \alpha) - L_1| < \frac{\varepsilon}{2}$$

Since L_2 satisfies the definition, there exists $\delta^2_{\varepsilon/2} : [a,b] \to (0,\infty)$ such that if P_2 is a $\delta^2_{\varepsilon/2}$-fine partition then

$$|S(f, P_2, \alpha) - L_2| < \frac{\varepsilon}{2}$$

We define $\delta_\varepsilon : [a,b] \to (0,\infty)$ by $\delta_\varepsilon(t) = \min\{\delta^1_{\varepsilon/2}(t), \delta^2_{\varepsilon/2}(t)\}$ for all $t \in [a,b]$. So δ_ε is a gauge on $[a,b]$ since the minimum of two positive numbers is still positive. If P is a δ_ε-fine partition, then P is both $\delta^1_{\varepsilon/2}$-fine and $\delta^2_{\varepsilon/2}$-fine: for all $i \in \{1,\ldots,n\}$ we have

$$t_i \in I_i \subseteq [t_i - \delta_\varepsilon(t_i), t_i + \delta_\varepsilon(t_i)] \subseteq [t_i - \delta^1_{\varepsilon/2}(t_i), t_i + \delta^1_{\varepsilon/2}(t_i)]$$

and

$$t_i \in I_i \subseteq [t_i - \delta_\varepsilon(t_i), t_i + \delta_\varepsilon(t_i)] \subseteq [t_i - \delta^2_{\varepsilon/2}(t_i), t_i + \delta^2_{\varepsilon/2}(t_i)]$$

Hence, from the definition, we have

$$|S(f, P, \alpha) - L_1| < \frac{\varepsilon}{2} \quad \text{and} \quad |S(f, P, \alpha) - L_2| < \frac{\varepsilon}{2}$$

Now we have

$$\begin{aligned}|L_1 - L_2| &= |(S(f, P, \alpha) - L_2) - (S(f, P, \alpha) - L_1)| \\ &\leq |(S(f, P, \alpha) - L_2)| + |(S(f, P, \alpha) - L_1)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $L_1 = L_2$, and so the value of the integral is uniquely determined. □

The next results prove that the collection of Generalised Riemann-Stieltjes integrable functions is a vector space.

Theorem 7. *If $f \in GRS([a, b], \alpha)$ and $k \in \mathbb{R}$ then $(kf) \in GRS([a, b], \alpha)$ and*

$$\int_a^b (kf) d\alpha = k \int_a^b f d\alpha$$

Proof. Fix $\varepsilon > 0$. Let $A := \int_a^b f d\alpha$. If $k = 0$ then $kA = 0$, and the function kf is identically zero so $\int_a^b kf = 0$ (by Theorem 1). If $k \neq 0$ then let $\delta_{\varepsilon/|k|}$ be a gauge such that if the partition $P = \{(I_i, t_i)\}_{i=1}^n$ is $\delta_{\varepsilon/|k|}$-fine then

$$|S(f, P, \alpha) - A| < \frac{\varepsilon}{|k|}$$

(This is possible since $f \in GRS([a, b], \alpha)$. We have $|k| > 0$ and so multiplying the above inequality by $|k|$ gives

$$|k||S(f, P, \alpha) - A| < \varepsilon$$

and therefore

$$|kS(f, P, \alpha) - kA| < \varepsilon$$

Now, we have

$$\begin{aligned} S(kf, P, \alpha) &= \sum_{i=1}^{n} (kf)(t_i)[\alpha(x_i) - \alpha(x_{i-1})] \\ &= \sum_{i=1}^{n} kf(t_i)[\alpha(x_i) - \alpha(x_{i-1})] \\ &= k\sum_{i=1}^{n} f(t_i)[\alpha(x_i) - \alpha(x_{i-1})] \\ &= kS(f, P, \alpha) \end{aligned}$$

So by substituting into (3) we obtain

$$|S(kf, P, \alpha) - kA| < \varepsilon$$

Hence $kf \in GRS([a,b], \alpha)$ and $\int_a^b (kf)d\alpha = kA = k\int_a^b f d\alpha$. □

Theorem 8. *If $f, g \in GRS([a,b], \alpha)$ then $f + g \in GRS([a,b], \alpha)$ and*

$$\int_a^b (f+g)d\alpha = \int_a^b f d\alpha + \int_a^b g d\alpha$$

Proof. Fix $\varepsilon > 0$. Let $A := \int_a^b f d\alpha$ and $B := \int_a^b g d\alpha$. Let $\delta_\varepsilon^1, \delta_\varepsilon^2 : [a,b] \to (0,\infty)$ be gauges such that if partition P_1 is δ_ε^1-fine and partition P_2 is δ_ε^2-fine then

$$|S(f, P_1, \alpha) - A| < \frac{\varepsilon}{2} \quad \text{and} \quad |S(f, P_2, \alpha) - B| < \frac{\varepsilon}{2}$$

We define $\delta_\varepsilon(t) := \min\{\delta_\varepsilon^1(t), \delta_\varepsilon^2(t)\}$ for all $t \in [a,b]$. If a partition $P = \{(I_i, t_i)\}_{i=1}^n$ is δ_ε-fine then it is also δ_ε^1-fine and δ_ε^2-fine. We have

$$\begin{aligned} S(f+g, P, \alpha) &= \sum_{i=1}^{n} (f+g)(t_i)[\alpha(x_i) - \alpha(x_{i-1})] \\ &= \sum_{i=1}^{n} f(t_i)[\alpha(x_i) - \alpha(x_{i-1})] + \sum_{i=1}^{n} g(t_i)[\alpha(x_i) - \alpha(x_{i-1})] \\ &= S(f, P, \alpha) + S(g, P, \alpha) \end{aligned}$$

and so, using the Triangle Inequality,

$$\begin{aligned}|S(f+g,P,\alpha)-(A+B)| &= |S(f,P,\alpha)-A+S(f,P,\alpha)-B| \\ &\leq |S(f,P,\alpha)-A|+|S(f,P,\alpha)-B| \\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} \\ &= \varepsilon\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $f, g \in GRS([a,b],\alpha)$ and

$$\int_a^b (f+g)d\alpha = A+B = \int_a^b f d\alpha + \int_a^b g d\alpha$$

□

Remark. Using the process of induction, it can be shown that if $f_1, \ldots, f_n \in GRS([a,b],\alpha)$ with $\int_a^b f_j d\alpha = L_j$ and $k_1, \ldots, k_n \in \mathbb{R}$ then $\sum_{j=1}^n k_j f_j \in GRS([a,b],\alpha)$ and

$$\int_a^b \sum_{j=1}^n k_j f_j d\alpha = \sum_{j=1}^n k_j L_j$$

So linear combinations of Generalised Riemann-Stieltjes integrable functions are also Generalised Riemann-Stieltjes integrable, and the integral of the sum is equal to the sum of the integrals.

The next results we see are important because they show that under certain conditions, integration respects inequalities.

Theorem 9. *If $\alpha : [a,b] \to \mathbb{R}$ is an increasing function and $f \in GRS([a,b],\alpha)$ with $f(x) \geq 0$ for all $x \in [a,b]$ then*

$$\int_a^b f \geq 0$$

Proof. Fix $\varepsilon > 0$ and define $A := \int_a^b f d\alpha$. Let $\delta_\varepsilon : [a,b] \to \mathbb{R}$ be a gauge such that if the partition $P = \{(I_i, t_i)\}_{i=1}^n$ is δ_ε-fine then $|S(f,P,\alpha) - A| < \varepsilon$.

Since $f(x) \geq 0$ for all $x \in I$, and α increasing implies $\alpha(x_i) - \alpha(x_{i-1}) \geq 0$ for all $i \in \{1, \ldots, n\}$ we have

$$S(f, P, \alpha) = \sum_{i=1}^{n} f(t_i)[\alpha(x_i) - \alpha(x_{i-1})] \geq 0$$

Hence $0 \leq S(f, P, \alpha) \leq A + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $A = \int_a^b f d\alpha \geq 0$. □

Corollary 1. *(Monotonicity) If $\alpha : [a, b] \to \mathbb{R}$ is an increasing function and $f, g \in GRS([a, b], \alpha)$ with $f(x) \leq g(x)$ for all $x \in [a, b]$ then*

$$\int_a^b f d\alpha \leq \int_a^b g d\alpha$$

Proof. We define the function $h : [a, b] \to \mathbb{R}$ by $h(x) := g(x) - f(x)$. By Theorems 7 and 8, we know that $h \in GRS([a, b], \alpha)$ and $\int_a^b h d\alpha = \int_a^b g d\alpha - \int_a^b f d\alpha$. Now, $f(x) \leq g(x)$ for all $x \in I$ implies that $h(x) \geq 0$ for all $x \in I$. Using Theorem 9, we have $\int_a^b h d\alpha \geq 0$ and therefore

$$\int_a^b g d\alpha - \int_a^b f d\alpha \geq 0 \Leftrightarrow \int_a^b f d\alpha \leq \int_a^b g d\alpha$$

□

The upcoming corollary is a consequence of the previous theorems and also the result concerning integration of a constant.

Corollary 2. *If $\alpha : [a, b] \to \mathbb{R}$ is an increasing function, $f \in GRS([a, b], \alpha)$, and $m, M \in \mathbb{R}$ with $m \leq f(x) \leq M$ for all $x \in I$ then*

$$m[\alpha(b) - \alpha(a)] \leq \int_a^b f d\alpha \leq M[\alpha(b) - \alpha(a)]$$

Proof. Firstly, note that using Theorem 1, we have $m, M \in GRS([a, b], \alpha)$. We have $f(x) - m \geq 0$ and so $\int_a^b (f - m) d\alpha \geq 0$. Using Theorem 8 we conclude that $\int_a^b f d\alpha - \int_a^b m d\alpha \geq 0$. So

$$\int_a^b f d\alpha \geq \int_a^b m d\alpha = m[\alpha(b) - \alpha(a)]$$

Similarly, since $M - f(x) \geq 0$ on I, we have $\int_a^b M d\alpha - \int_a^b f d\alpha \geq 0$ and so

$$\int_a^b f d\alpha \leq \int_a^b M d\alpha = M[\alpha(b) - \alpha(a)]$$

□

This corollary clarifies the interaction between the integral and the absolute value function.

Corollary 3. *If $\alpha : [a, b] \to \mathbb{R}$ is an increasing function, and $f, |f| \in GRS([a, b], \alpha)$ then*

$$\left| \int_a^b f d\alpha \right| \leq \int_a^b |f| d\alpha$$

Proof. We have

$$-|f(x)| \leq f(x) \leq |f(x)| \quad \forall x \in I$$

and so, using monotonicity,

$$-\int_a^b |f| d\alpha \leq \int_a^b f d\alpha \leq \int_a^b |f| d\alpha$$

Contracting the absolute value inequality, we conclude that

$$\left| \int_a^b f d\alpha \right| \leq \int_a^b |f| d\alpha$$

□

We recall that a Cauchy sequence is one in which the terms become closer and closer to each other rather than to a particular limit. The next result aims to rewrite the definition of integrability in a similar form - by assuming that the Riemann-Stieltjes sums are “close” to each other rather than to a number (the integral). It successfully characterises integrability in these terms.

Theorem 10. *(Cauchy Criterion) Let $\alpha : I \to \mathbb{R}$ be an integrator function. For a function $f : I \to \mathbb{R}$, we have $f \in GRS(I, \alpha)$ if and only if for all $\varepsilon > 0$ there exists a gauge $\gamma : I \to (0, \infty)$ such that if P, Q are two γ-fine partitions then*

$$|S(f, P, \alpha) - S(f, Q, \alpha)| < \varepsilon$$

Proof. For ($\Rightarrow$): If $f \in GRS(I, \alpha)$ with integral A, then let $\delta_{\varepsilon/2}$ be a gauge on I such that if P, Q are $\delta_{\varepsilon/2}$-fine then

$$|S(f, P, \alpha) - A| < \frac{\varepsilon}{2} \quad \text{and} \quad |S(f, Q, \alpha) - A| < \frac{\varepsilon}{2}$$

Now, using the Triangle Inequality, we have

$$\begin{aligned} |S(f, P, \alpha) - S(f, Q, \alpha)| &= |(S(f, P, \alpha) - A) + (A - S(f, Q, \alpha))| \\ &\leq |S(f, P, \alpha) - A| + |S(f, Q, \alpha) - A| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

as required.

For ($\Leftarrow$): For each $n \in \mathbb{N}$, let $\delta_n : I \to (0, \infty)$ be a gauge on I such that if P, Q are δ_n-fine partitions then

$$|S(f, P, \alpha) - S(f, Q, \alpha)| < \frac{1}{n}$$

Without loss of generality, we may assume that $\delta_n(t) \geq \delta_{n+1}(t)$ for all $t \in I$, as otherwise we may replace δ_n by $\delta'_n = \min\{\delta_1, \ldots, \delta_n\}$. Now for each $n \in \mathbb{N}$, let P_n be a tagged partition which is δ_n-fine. If $m > n$ then both P_n and P_m are δ_n-fine, and so we have

$$|S(f, P_n, \alpha) - S(f, P_m, \alpha)| < \frac{1}{n} \tag{3.1}$$

Hence, the sequence of Riemann-Stieltjes sums $\{S(f, P_m, \alpha)\}_{m=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, the sequence must also be convergent. We define

$$A = \lim_{m\to\infty} S(f, P_m, \alpha)$$

so that after taking limits in (3.1) we obtain

$$\lim_{m\to\infty} |S(f, P_n, \alpha) - S(f, P_m, \alpha)| < \lim_{m\to\infty} \frac{1}{n}$$

and therefore

$$|S(f, P_n, \alpha) - A| < \frac{1}{n}$$

We will now show that $A = \int_a^b f d\alpha$. Fix $\varepsilon > 0$ and let $K \in \mathbb{N}$ be such that $K > \frac{2}{\varepsilon}$. If Q is a δ_K fine partition then using the Triangle Inequality we have

$$\begin{aligned} |S(f, Q, \alpha) - A| &= |(S(f, Q, \alpha) - S(f, P_K, \alpha) + S(f, P_K, \alpha) - A| \\ &\leq |S(f, Q, \alpha) - S(f, P_K, \alpha)| + |S(f, P_K, \alpha) - A| \\ &\leq \frac{1}{K} + \frac{1}{K} \\ &= \frac{2}{K} \\ &< \varepsilon \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $f \in GRS(I, \alpha)$ with $\int_a^b f d\alpha = A$. □

We are familiar with the Squeeze Theorem or Sandwich Theorem for convergent sequences. The next result uses a similar technique to characterise integrability of a function f by examining the integrability of two other suitable functions which are hopefully more computationally amenable.

Theorem 11. *(Squeeze Theorem) The function $f \in GRS(I, \alpha)$ if and only if for all $\varepsilon > 0$ there exist functions $\phi_\varepsilon, \psi_\varepsilon \in GRS(I, \alpha)$ with $\phi_\varepsilon(x) \leq f(x) \leq \psi_\varepsilon(x)$ for all $x \in I$ and such that $\int_I (\psi_\varepsilon - \phi_\varepsilon) d\alpha < \varepsilon$.*

Proof. For ($\Rightarrow$): Fix $\varepsilon > 0$. If $f \in GRS(I, \alpha)$, then take $\phi_\varepsilon = \psi_\varepsilon = f$, and so $\int_I (\psi_\varepsilon - \phi_\varepsilon) d\alpha = \int_I 0 d\alpha = 0 < \varepsilon$.

For ($\Leftarrow$): Fix $\varepsilon > 0$. Then for any tagged partition P of I we have

$$\phi_\varepsilon \leq f \leq \psi_\varepsilon \Rightarrow S(\phi_\varepsilon, P, \alpha) \leq S(f, P, \alpha) \leq S(\psi_\varepsilon, P, \alpha)$$

Now since $\phi_\varepsilon \in GRS(I, \alpha)$, there exists a gauge $\delta^1_\varepsilon : I \to (0, \infty)$ such that if P is a δ^1_ε-fine partition then $|S(\phi_\varepsilon, P, \alpha) - \int_I \phi_\varepsilon d\alpha| < \varepsilon$. By rearranging we obtain

$$\int_I \phi_\varepsilon d\alpha - \varepsilon \leq S(\phi_\varepsilon, P, \alpha)$$

Since $\psi_\varepsilon \in GRS(I, \alpha)$, there exists a gauge $\delta^2_\varepsilon : I \to (0, \infty)$ such that if P is a δ^2_ε-fine partition then $|S(\psi_\varepsilon, P, \alpha) - \int_I \psi_\varepsilon d\alpha| < \varepsilon$. By rearranging we obtain

$$S(\phi_\varepsilon, P, \alpha) \leq \int_I \phi_\varepsilon d\alpha + \varepsilon$$

We define the gauge $\delta_\varepsilon := \min\{\delta^1_\varepsilon, \delta^2_\varepsilon\}$. So if P is δ_ε-fine then P is both δ^1_ε-fine and δ^2_ε-fine, and

$$\int_I \phi_\varepsilon d\alpha - \varepsilon \leq S(\phi_\varepsilon, P, \alpha) \leq S(f, P, \alpha) \leq S(\psi_\varepsilon, P, \alpha) \leq \int_I \psi_\varepsilon + \varepsilon$$

Similarly, if Q is δ_ε-fine then

$$-\int_I \psi_\varepsilon - \varepsilon \leq -S(f, Q, \alpha) \leq -\int_I \phi_\varepsilon + \varepsilon$$

If we add the two inequalities we obtain

$$-\int_I (\psi_\varepsilon - \phi_\varepsilon) d\alpha - 2\varepsilon \leq S(f, P, \alpha) - S(f, Q, \alpha) \leq \int_I (\psi_\varepsilon - \phi_\varepsilon) d\alpha + 2\varepsilon$$

Hence, from the definition of absolute value,

$$|S(f, P, \alpha) - S(f, Q, \alpha)| \leq \int_I (\psi_\varepsilon - \phi_\varepsilon) d\alpha + 2\varepsilon < 3\varepsilon$$

using the condition that $\int_I (\psi_\varepsilon - \phi_\varepsilon) d\alpha < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, f satisfies the Cauchy Criterion, and hence f is integrable on I. □

Thus far, we have only considered integrals on a single interval, and have been more interested in the integrand and integrator than the interval itself, however, the next result aims to relate integrals of the same functions on different intervals.

Theorem 12. *(Additivity Theorem) Let $\alpha : [a,b] \to \mathbb{R}$ be an integrator function. Suppose that $f : [a,b] \to \mathbb{R}$ and $c \in (a,b)$. Then $f \in GRS([a,b],\alpha)$ if and only if the restrictions $f|_{[a,c]}$ and $f|_{[c,b]}$ are Generalised Riemann-Stieltjes integrable with respect to α. In this case, we have*

$$\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha$$

Proof. For ($\Leftarrow$): Suppose that $f_1 = f|_{[a,c]}$ is integrable to L_1 and $f_2 = f|_{[c,b]}$ is integrable to L_2. Fix $\varepsilon > 0$. There exists a gauge $\delta^1 : [a,c] \to (0,\infty)$ such that if P_1 is a δ^1-fine partition of $[a,c]$ then

$$|S(f_1, P_1, \alpha) - L_1| < \frac{\varepsilon}{2}$$

Similarly, there exists a gauge $\delta^2 : [c,b] \to (0,\infty)$ such that if P_2 is a δ^2-fine partition of $[c,b]$ then

$$|S(f_2, P_2, \alpha) - L_2| < \frac{\varepsilon}{2}$$

We define a gauge $\delta_\varepsilon : [a,b] \to (0,\infty)$ by

$$\delta(x) = \begin{cases} \min\{\delta^1(x), \frac{1}{2}(c-x)\} & \text{if} \quad x \in [a,c) \\ \min\{\delta^1(x), \delta^2(x)\} & \text{if} \quad x = c \\ \min\{\delta^2(x), \frac{1}{2}(x-c)\} & \text{if} \quad x \in (c,b] \end{cases}$$

Claim: Any δ_ε-fine partition of $[a,b]$ has c as a tag for any subinterval containing c.

Proof of Claim: Suppose $I_i = [x_{i-1}, x_i]$ is an interval in a δ_ε-fine partition, and that $c \in I_i$. We will assume by way of a contradiction that the tag $t_i \neq c$. Suppose that $t_i < c$. Since $c \in I_i$ and $t_i \in I_i$, we know that the length of I_i is greater than or equal to $c - t_i$. Now, $\delta_\varepsilon(t_i) = \min\{\delta^1(t_i), \frac{1}{2}(c-t_i)\} \leq \frac{1}{2}(c-t_i)$. So by δ_ε-fineness we have

$$\begin{aligned} [x_{i-1}, x_i] &\subseteq [t_i - \delta_\varepsilon(t_i), t_i + \delta_\varepsilon(t_i)] \\ &\subseteq [t_i - \frac{1}{2}(c - t_i), t_i + \frac{1}{2}(c - t_i)] \end{aligned} \tag{3.2}$$

So the length of I_i is less than or equal to $c - t_i$. Hence the length must be $c - t_i$, and therefore $[x_{i-1}, x_i] = [t_i, c]$. Substituting into (3.2), we obtain

$$[t_i, c] \subseteq [t_i - \frac{1}{2}(c - t_i), t_i + \frac{1}{2}(c - t_i)]$$

and therefore we must have

$$c \leq t_i + \frac{1}{2}(c - t_i) = \frac{t_i}{2} + \frac{c}{2}$$

But since $t_i < c$ we have $\frac{t_i}{2} < \frac{c}{2}$ and therefore $c > \frac{t_i}{2} + \frac{c}{2}$. This is a contradiction. The argument is similar for the case of $t_i > c$. So we conclude that $t_i = c$ as required.

Claim: If Q is any δ_ε-fine partition of $[a, b]$ then there exists a δ^1-fine partition Q_1 of $[a, c]$ and a δ^2-fine partition Q_2 of $[c, b]$ such that

$$S(f, Q, \alpha) = S(f_1, Q_1, \alpha) + S(f_2, Q_2, \alpha)$$

Proof of Claim: We will work by cases.

Case 1: c is a partition point of Q. Then c is contained in two subintervals of Q and, using the previous claim, is the tag for both of them. Let us define Q_1 to be the set of pairs in Q with subintervals in $[a, c]$, and Q_2 to be the set of pairs in Q_2 with subintervals in $[c, b]$. Then Q_1 is δ^1-fine and Q_2 is δ^2-fine, and we have

$$S(f, Q, \alpha) = S(f_1, Q_1, \alpha) + S(f_2, Q_2, \alpha)$$

Case 2: c is not a partition point of Q. Then c is the tag for precisely one subinterval of Q, let's say $[x_{k-1}, x_k]$. We will replace the pair $([x_{k-1}, x_k], c)$ by the two pairs $([x_{k-1}, c], c)$ and $([c, x_k], c)$. Let Q_1 and Q_2 be the tagged partitions of $[a, c]$ and $[c, b]$ respectively that result from this procedure. We examine the contribution to the Riemann-Stieltjes sum

$$f(c)(\alpha(x_k) - \alpha(x_{k-1})) = f(c)[\alpha(c) - \alpha(x_{k-1})] + f(c)[\alpha(x_k) - \alpha(c)]$$

Hence

$$S(f, Q, \alpha) = S(f_1, Q_1, \alpha) + S(f_2, Q_2, \alpha)$$

In both case 1 and case 2, we have, via the Triangle Inequality,

$$\begin{aligned} |S(f, Q, \alpha) - (L_1 + L_2)| &= |S(f_1, Q_1, \alpha) + S(f_2, Q_2, \alpha) - (L_1 + L_2)| \\ &\leq |S(f_1, Q_1, \alpha) - L_1| + |S(f_2, Q_2, \alpha) - L_2| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned} \tag{3.3}$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $f \in GRS([a, b], \alpha)$ and

$$\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha$$

For ($\Rightarrow$): Suppose that $f \in GRS([a, b], \alpha)$. Fix $\varepsilon > 0$, and let γ_ε be a gauge which satisfies the Cauchy criterion. We define $f_1 := f|_{[a,c]}$, and let P_1, Q_1 be γ_ε-fine partitions of $[a, c]$. Through the process of adding partition points and tags appropriately, we can extend P_1 and Q_1 to γ_ε-fine partitions P and Q of $[a, b]$. By adding the same partition points and tags to both P_1 and Q_1, we have

$$S(f, P, \alpha) - S(f, Q, \alpha) = S(f_1, P_1, \alpha) - S(f_1, Q_1, \alpha)$$

Now since P and Q are γ_ε-fine we have $|S(f_1, P_1, \alpha) - S(f_1, Q_1, \alpha)| < \varepsilon$. Using the Cauchy Criterion, we conclude that $f_1 \in GRS([a, b], \alpha)$. A similar argument can be used to show that $f_2 \in GRS([a,b],\alpha)$. Now, applying the result from the ($\Leftarrow$) part of this argument, we conclude that

$$\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha$$

$\square$

Corollary 4. *If $f \in GRS([a,b],\alpha)$ and $[c,d] \subseteq [a,b]$ then the restriction $f_{[c,d]}$ is Generalised Riemann-Stieltjes integrable.*

Proof. We apply the Additivity Theorem twice: Since f is integrable on $[a,b]$ and $c \in [a,b]$, the restriction $f_{[c,b]}$ is integrable. Now $d \in [c,b]$, and so the restriction $f_{[c,d]}$ is integrable. $\square$

Remark. Using the Additivity Theorem and induction, it can be shown that: If $f \in GRS([a,b],\alpha)$ and $a = c_0 < c_1 < \cdots < c_n = b$ then the restriction $f_{[c_{i-1},c_i]}$ is integrable and

$$\int_a^b f d\alpha = \sum_{i=1}^{n} \int_{c_{i-1}}^{c_i} f d\alpha$$

Finally, as a way of clarifying some technical details, we have the following definition.

Definition 14. For $\alpha, f : I \to \mathbb{R}$ and $f \in GRS(I,\alpha)$ we define

$$\int_a^a f d\alpha := 0 \quad \text{and} \quad \int_b^a f d\alpha := -\int_a^b f d\alpha$$

Chapter 4

The α-derivative

In this section, we consider a little known type of derivative which is motivated by the requirement of an "inverse" for the Generalised Riemann-Stieltjes integral. Since we are integrating with respect to a function α, it also makes sense to consider differentiating with respect to it, however, this derivative is much less popular than the corresponding integral. Perhaps part of the reason for this is because, as we will soon see, in many cases it simply reduces down to a formula involving ordinary derivatives. Nevertheless, the α-derivative is both interesting in its own right and an essential element of the proof of the Fundamental Theorem of Calculus for the Riemann-Stieljes and Generalised Riemann-Stieltjes integrals, as we will see in the next section.

Definition 15. Suppose $f, \alpha : I \to \mathbb{R}$ are functions such that α is continuous and strictly increasing. Suppose $x_0 \in I$. We say that f is α-differentiable at x_0 if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{\alpha(x) - \alpha(x_0)}$$

exists. If this limit exists then we denote its value by $D_\alpha f(x_0)$. We say that f is α-differentiable on I if f is α-differentiable at every point of I.

Remark. Note that if $\alpha(x) = x$ then the α-derivative reduces to the usual derivative.

The requirements for α to be continuous and strictly increasing are not totally necessary, however, their presence in the definition is due to the requirement for these conditions in some of the proofs of desirable properties, as we will soon see.

Remark. To emphasise the point, the α-derivative is the correct derivative to use when working with Riemann-Stieltjes or Generalised Riemann-Stieltjes integrals; since there is an α in the integral it makes sense for it to also be a part of the derivative.

Note that Definition 15 is equivalent to the following definition:

Definition 16. Suppose $f, \alpha : I \to \mathbb{R}$ are functions such that α is continuous and strictly increasing. Suppose $x_0 \in I$. We say that f is α-differentiable at x_0 if

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{\alpha(x+h) - \alpha(x)}$$

exists.

The following result is important because it relates the ordinary and α derivatives to each other.

Theorem 13. *Let $f, \alpha : I \to \mathbb{R}$ be functions such that α is continuous and strictly increasing. Suppose $x_0 \in I$. If the ordinary derivatives $f'(x_0)$ and $\alpha'(x_0)$ both exist, and $\alpha'(x_0) \neq 0$ then*

$$D_\alpha f(x_0) = \frac{f'(x_0)}{\alpha'(x_0)}$$

Proof. We have

$$\begin{aligned} D_\alpha f(x_0) &= \lim_{x\to x_0} \frac{f(x)-f(x_0)}{\alpha(x)-\alpha(x_0)} \\ &= \lim_{x\to x_0} \frac{[f(x)-f(x_0)]/(x-x_0)}{[\alpha(x)-\alpha(x_0)]/(x-x_0)} \\ &= \frac{f'(x_0)}{\alpha'(x_0)} \end{aligned}$$

□

Remark. Under the conditions of the previous theorem, the α-derivative is simply a rescaling of the ordinary derivative. This provides a geometric interpretation.

Example 10. Suppose $[a, b] = [0, 1]$, $f(x) = x^4$ and $\alpha(x) = x^2$. We have

$$D_\alpha f(x_0) = \frac{f'(x_0)}{\alpha'(x_0)} = \frac{4x_0^3}{2x_0} = 2x_0^2$$

At the point $x_0 = 0.5$, we have $\alpha'(x_0) = 1$ and so there is no rescaling taking place; the α-derivative is equal to the ordinary derivative.

Compare this to the point $x_0 = \frac{1}{3}$, where $\alpha'(x_0) = \frac{2}{3}$. We compare the gradients of the two tangent lines: the ordinary derivative is $f'(x_0) = \frac{4}{9}$, whereas the α-derivative is $D_\alpha f(x_0) = \frac{4}{9} \div \frac{2}{3} = \frac{2}{3}$.

Example 11. If $f(x) = c$ where $c \in R$ for all $x \in I$ then

$$D_\alpha f(x_0) = \lim_{x\to x_0} \frac{c-c}{\alpha(x)-\alpha(x_0)} = 0$$

If $f(x) = \alpha(x)$ for all $x \in I$ then

$$D_(x_0) = \lim_{x\to x_0} \frac{\alpha(x)-\alpha(x_0)}{\alpha(x)-\alpha(x_0)} = 1$$

If $f(x) = x^p$ and $\alpha(x) = x^q$ for $p \in \mathbb{R}, q \neq 0 \in \mathbb{R}$ then for $x_0 \neq 0$ we have

$$D_\alpha f(x_0) = \frac{f'(x_0)}{\alpha'(x_0)} = \frac{px_0^{p-1}}{qx_0^{q-1}} = \frac{p}{q}x_0^{p-q}$$

and for $x_0 = 0$, if $0 \in I$, we have $D_\alpha f(0) = 0$, therefore $D_\alpha f(x_0) = \frac{p}{q}x_0^{p-q}$.

In the next original example, we see how the previous theorem can be used to solve α-differential equations.

Example 12. Suppose we have a DE of the form

$$D_\alpha y(x) - ky(x) = 0$$

for $k \in R$ and suitable y, α. Then applying the rescaling, we have

$$\frac{y'(x)}{\alpha'(x)} - ky(x) = 0$$

and so we obtain the familiar form

$$y'(x) = k\alpha'(x)y(x)$$

In the following original examples, we see that there are functions f for which $f'(x)$ doesn't exist but $D_\alpha f(x)$ does for some nontrivial $\alpha(x) \neq x$ (nontrivial meaning that $\alpha(x) \neq f(x)$, since $D_f f(x)$ always exists). It is also an example of the use of the α-derivative for functions f, α which are not differentiable at a point, which shows that the previous theorem cannot be used for all α-derivatives.

Example 13. Note that there are some functions $\alpha : [a, b] \to \mathbb{R}$ which are strictly increasing and continuous but not differentiable, so the previous theorem is not true for general α. Consider the function $\alpha : [0, 2] \to \mathbb{R}$ given by

$$\alpha(x) = \begin{cases} x & \text{if} \quad x \in [0,1] \\ x^2 & \text{if} \quad x \in [1,2] \end{cases}$$

Then α is continuous and strictly increasing, but not differentiable at $x = 1$. So the previous theorem cannot be applied to the α-derivative of any function $f : [0, 2] \to \mathbb{R}$.

Example 14. Suppose $f, \alpha : [-1, 1] \to \mathbb{R}$ are given by $f(x) = |x|$ and

$$\alpha(x) = \begin{cases} -\sqrt{-x} & \text{if} \quad x \in [-1, 0] \\ \sqrt{x} & \text{if} \quad x \in (0, 1] \end{cases}$$

Now, we know that $f'(x)$, the ordinary derivative, does not exist at $x = 0$, but we will show using first principles that $D_\alpha f(x)$ does exist at $x = 0$. We consider the limits from the left and right:

$$\begin{aligned} \lim_{x \to 0^+} \frac{f(x) - f(0)}{\alpha(x) - \alpha(0)} &= \lim_{x \to 0^+} \frac{x}{\sqrt{x}} \\ &= \lim_{x \to 0^+} \sqrt{x} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \lim_{x \to 0^-} \frac{f(x) - f(0)}{\alpha(x) - \alpha(0)} &= \lim_{x \to 0^-} \frac{-x}{-\sqrt{-x}} \\ &= \lim_{x \to 0^-} \sqrt{-x} \\ &= 0 \end{aligned}$$

Since the limits are equal, $D_\alpha f(0)$ exists.

Remark. Note that the previous example can be extended to functions of the more general form $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = |kx - a|$ for $k > 0, a \in \mathbb{R}$. We choose

$$\alpha(x) = \begin{cases} \sqrt{kx - a} & \text{if} \quad x \geq \frac{a}{k} \\ -\sqrt{-(kx - a)} & \text{if} \quad x < \frac{a}{k} \end{cases}$$

The following theorem proves the continuity of α-differentiable functions.

Theorem 14. *If $\alpha : I \to \mathbb{R}$ is continuous and strictly increasing, and $f : I \to \mathbb{R}$ is α-differentiable at x_0, then f is continuous at x_0.*

Proof. We have

$$\begin{aligned}\lim_{x\to x_0}[f(x)-f(x_0)] &= \lim_{x\to x_0}\frac{f(x)-f(x_0)}{\alpha(x)-\alpha(x_0)}[\alpha(x)-\alpha(x_0)]\\ &= D_\alpha f(x_0)\cdot 0\\ &= 0\end{aligned}$$

where we can use the algebra of limits since both limits exist, and since α is continuous we have $\lim_{x\to x_0}[\alpha(x)-\alpha(x_0)]=0$. □

The next few results are analogues of theorems concerning the ordinary derivative, and these proofs are based on the standard proofs of those results, with some minor modifications. As in the more familiar context, they provide helpful tools for calculation of more complicated derivatives.

Theorem 15. *(Algebra of α-differentiable functions) If $\alpha : I \to \mathbb{R}$ is continuous and strictly increasing, and $f, g : I \to \mathbb{R}$ are α-differentiable x_0 then $f+g, fg, cf$ and $\frac{f}{g}$ are α-differentiable at x_0 and*

1. $D_\alpha(f+g)(x_0) = D_\alpha f(x_0) + D_\alpha g(x_0)$
2. $D_\alpha(fg)(x_0) = [D_\alpha f(x_0)]g(x_0) + f(x_0)[D_\alpha g(x_0)]$
3. $D_\alpha(cf)(x_0) = cD_\alpha f(x_0)$
4. $D_\alpha\left(\frac{f}{g}\right)(x_0) = \frac{[D_\alpha f(x_0)]g(x_0)-f(x_0)[D_\alpha g(x_0)]}{g(x_0)^2}$ *if* $g(x_0)\neq 0$

Proof. (1): We have

$$\begin{aligned}
D_\alpha(f+g)(x_0) &= \lim_{x\to x_0} \frac{(f+g)(x)-(f+g)(x_0)}{\alpha(x)-\alpha(x_0)} \\
&= \lim_{x\to x_0} \frac{f(x)+g(x)-f(x_0)-g(x_0)}{\alpha(x)-\alpha(x_0)} \\
&= \lim_{x\to x_0} \left[\left(\frac{f(x)-f(x_0)}{\alpha(x)-\alpha(x_0)}\right)+\left(\frac{g(x)-g(x_0)}{\alpha(x)-\alpha(x_0)}\right)\right] \\
&= \lim_{x\to x_0} \frac{f(x)-f(x_0)}{\alpha(x)-\alpha(x_0)} + \lim_{x\to x_0} \frac{g(x)-g(x_0)}{\alpha(x)-\alpha(x_0)} \\
&= D_\alpha f(x_0) + D_\alpha g(x_0)
\end{aligned}$$

We can use the algebra of limits since f, g are α-differentiable, and hence both limits exist.

(2): We have

$$\begin{aligned}
D_\alpha(fg)(x_0) &= \lim_{x\to x_0} \frac{f(x)g(x)-f(x_0)g(x_0)}{\alpha(x)-\alpha(x_0)} \\
&= \lim_{x\to x_0} \frac{f(x)g(x)-f(x_0)g(x_0)+f(x)g(x_0)-f(x)g(x_0)}{\alpha(x)-\alpha(x_0)} \\
&= \lim_{x\to x_0} \left[\frac{f(x)[g(x)-g(x_0)]}{\alpha(x)-\alpha(x_0)} + \frac{g(x_0)[f(x)-f(x_0)]}{\alpha(x)-\alpha(x_0)}\right] \\
&= f(x_0)D_\alpha g(x_0) + g(x_0)D_\alpha f(x_0)
\end{aligned}$$

Again it is possible to use the algebra of limits since f, g are α-differentiable.

(3): Let $g(x) = c$ in part (2).

(4): We show that if g is α-differentiable at x_0 and $g(x_0) \neq 0$ then $\frac{1}{g}$ is α-differentiable at x_0. We have

$$\begin{aligned}
\frac{\frac{1}{g(x)} - \frac{1}{g(x_0)}}{\alpha(x)-\alpha(x_0)} &= \frac{\frac{g(x_0)-g(x)}{g(x)g(x_0)}}{\alpha(x)-\alpha(x_0)} \\
&= -\frac{g(x)-g(x_0)}{\alpha(x)-\alpha(x_0)}\frac{1}{g(x)g(x_0)}
\end{aligned}$$

Now, since g is α-differentiable at x_0, the limit $\lim_{x\to x_0} \frac{g(x)-g(x_0)}{\alpha(x)-\alpha(x_0)}$ exists, and so the limit $\lim_{x\to x_0} -\frac{g(x)-g(x_0)}{\alpha(x)-\alpha(x_0)}$ exists and is equal to $-D_\alpha g(x_0)$. Since g is α-differentiable and therefore continuous at x_0, we have $\lim_{x\to x_0} g(x) = g(x_0)$, and therefore

$$\lim_{x \to x_0} \frac{1}{g(x)} = \frac{1}{g(x_0)}$$

since $g(x_0) \neq 0$. So

$$\lim_{x \to x_0} \frac{1}{g(x)g(x_0)} = \frac{1}{g(x_0)^2}$$

Hence, by the algebra of limits, we have

$$D_\alpha \left(\frac{1}{g}\right)(x_0) = \lim_{x \to x_0} -\frac{g(x) - g(x_0)}{\alpha(x) - \alpha(x_0)} \frac{1}{g(x)g(x_0)} = -\frac{D_\alpha g(x_0)}{g(x_0)^2}$$

So, applying (2), we obtain

$$D_\alpha \left(\frac{f}{g}\right)(x_0) = \frac{[D_\alpha f(x_0)]g(x_0) - f(x_0)[D_\alpha g(x_0)]}{g(x_0)^2}$$

□

The next result is an original theorem concerning the relationship between derivatives of the same function with respect to different α's.

Theorem 16. *Suppose $\alpha, \beta : I \to \mathbb{R}$ are strictly increasing, continuous functions. Suppose that $f : I \to \mathbb{R}$ is β-differentiable at x_0 and β is α-differentiable at x_0. Then f is α-differentiable at x_0 and*

$$D_\alpha f(x_0) = D_\beta f(x_0) D_\alpha \beta(x_0)$$

Proof. Since f is β-differentiable at x_0, the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{\beta(x) - \beta(x_0)}$$

exists. Since β is α-differentiable at x_0, the limit

$$\lim_{x \to x_0} \frac{\beta(x) - \beta(x_0)}{\alpha(x) - \alpha(x_0)}$$

exists. By the algebra of limits, we have

$$\lim_{x \to \infty} \frac{f(x) - f(x_0)}{\alpha(x) - \alpha(x_0)} = \left(\lim_{x \to x_0} \frac{f(x) - f(x_0)}{\beta(x) - \beta(x_0)}\right) \times \left(\lim_{x \to x_0} \frac{\beta(x) - \beta(x_0)}{\alpha(x) - \alpha(x_0)}\right)$$

and so

$$D_\alpha f(x_0) = D_\beta f(x_0) D_\alpha \beta(x_0)$$

□

The next result is another analogue of a theorem concerning the ordinary derivative. Again it is based on the standard proof of this result for ordinary derivatives, with some small changes.

Theorem 17. *(Chain Rule) Suppose that $\alpha : I \to \mathbb{R}$ is a strictly increasing, continuous function. Suppose that $g : I \to \mathbb{R}$ is α-differentiable at x, and $f : I \to \mathbb{R}$ is α-differentiable at $g(x)$. Then $f \circ g$ is differentiable at x and*

$$D_\alpha(f \circ g)(x) = D_\alpha(f)[g(x)] D_\alpha g(x)$$

Proof. We will use Definition 16 in this proof. Since g is α-differentiable at x, we have

$$\frac{g(x+h) - g(x)}{\alpha(x+h) - \alpha(x)} - D_\alpha g(x) \to 0 \quad \text{as} \quad h \to 0$$

We define

$$v := \frac{g(x+h) - g(x)}{\alpha(x+h) - \alpha(x)} - D_\alpha g(x)$$

So $v = v(h)$ is a function of h, and $v \to 0$ as $h \to 0$. Similarly, since f is α-differentiable at $y := g(x)$, we have

$$\frac{f(y+k) - f(y)}{\alpha(y+k) - \alpha(y)} - D_\alpha f(y) \to 0 \quad \text{as} \quad k \to 0$$

We define

$$w := \frac{f(y+k) - f(y)}{\alpha(y+k) - \alpha(y)} - D_\alpha f(y)$$

So $w = w(k)$ is a function of k, and $w \to 0$ as $k \to 0$. By rearranging, we obtain

$$g(x+h) = g(x) + [D_\alpha g(x) + v][\alpha(x+h) - \alpha(x)]$$

and

$$f(y+k) = f(y) + [D_\alpha f(y) + w][\alpha(y+k) - \alpha(y)]$$

So we can rewrite $f(g(x+h))$ as

$$\begin{aligned} f(g(x+h)) &= f[g(x) + [D_\alpha g(x) + v][\alpha(x+h) - \alpha(x)]] \\ &= f(g(x)) + [D_\alpha(f)(g(x)) + w][D_\alpha g(x) + v][\alpha(x+h) - \alpha(x)] \end{aligned}$$

So we have

$$\begin{aligned} &\frac{f(g(x+h)) - f(g(x))}{\alpha(x+h) - \alpha(x)} \\ =&\frac{f(g(x)) + [D_\alpha(f)(g(x)) + w][D_\alpha g(x) + v][\alpha(x+h) - \alpha(x)]}{\alpha(x+h) - \alpha(x)} \\ &- \frac{f(g(x))}{\alpha(x+h) - \alpha(x)} \\ =&\frac{[D_\alpha(f)(g(x)) + w][D_\alpha g(x) + v][\alpha(x+h) - \alpha(x)]}{\alpha(x+h) - \alpha(x)} \\ =&[D_\alpha(f)(g(x)) + w][D_\alpha g(x) + v] \end{aligned}$$

and therefore

$$\begin{aligned} &\lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{\alpha(x+h) - \alpha(x)} \\ =& \lim_{h\to 0} [D_\alpha(f)(g(x)) + w][D_\alpha g(x) + v] \\ =& \left(\lim_{h\to 0} D_\alpha(f)(g(x)) + \lim_{h\to 0} w\right)\left(\lim_{h\to 0} D_\alpha g(x) + \lim_{h\to 0} v\right) \\ =& D_\alpha f(g(x)) D_\alpha(g(x)) \end{aligned}$$

Note that we could use the algebra of limits since each separate limit exists. Recall that we have $v \to 0$ as $h \to 0$ and $w \to 0$ as $h \to 0$. □

The next results concern the classification of maxima, minima and turning points.

Theorem 18. *Suppose $\alpha : I \to \mathbb{R}$ is a strictly increasing continuous function, and $f : I \to \mathbb{R}$ is α-differentiable on (a, b). If f has a relative maximum or minimum at $x_0 \in (a, b)$ then $D_\alpha f(x_0) = 0$.*

Proof. Suppose that x_0 is a maximum. Then there exists $\delta > 0$ such that

$$a < x_0 - \delta < x_0 < x_0 + \delta < b \Rightarrow f(x_0) \geq f(x)$$

If $x_0 - \delta < x < x_0$ then we have

$$\frac{f(x) - f(x_0)}{\alpha(x) - \alpha(x_0)} \geq 0$$

since $f(x) - f(x_0) \leq 0$ and $\alpha(x) - \alpha(x_0) < 0$ since α is strictly increasing. Now let $x \to x_0$ from below. We have $D_\alpha f(x_0) \geq 0$. If $x_0 < x < x_0 + \delta$ then we have

$$\frac{f(x) - f(x_0)}{\alpha(x) - \alpha(x_0)} \leq 0$$

since $f(x) - f(x_0) \leq 0$ and $\alpha(x) - \alpha(x_0) > 0$. Now let $x \to x_0$ from above. We have $D_\alpha f(x_0) \leq 0$. So we must have $D_\alpha f(x_0) = 0$. The argument is similar when x_0 is a minimum. □

Theorem 19. *(Rolle's Theorem) Suppose $\alpha : I \to \mathbb{R}$ is a strictly increasing continuous function, and $f : I \to \mathbb{R}$ is continuous on $[a, b]$ and α-differentiable on (a, b). If $f(a) = f(b)$ then there exists $x_0 \in (a, b)$ such that $D_\alpha f(x_0) = 0$.*

Proof. If the minimum and maximum of f both occur at endpoints of the interval then f is constant and so $D_\alpha f(x_0) = 0$ for all $x_0 \in (a, b)$. Otherwise, either a minimum or a maximum occurs at $x_0 \in (a, b)$ and so $D_\alpha f(x_0) = 0$. □

The next theorems are analgoues to the Mean Value type theorems for the ordinary derivatives.

Theorem 20. *(Cauchy Mean Value Theorem) Suppose $\alpha : I \to \mathbb{R}$ is a strictly increasing continuous function, and $f, g : I \to \mathbb{R}$ are continuous* *functions which are α-differentiable on (a, b). Then there exists $x_0 \in (a, b)$ such that*

$$[f(b) - f(a)]D_\alpha g(x_0) = [g(b) - g(a)]D_\alpha f(x_0)$$

Proof. We apply the previous theorem to the function

$$h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x)$$

. □

Theorem 21. *(Mean Value Theorem) Suppose $\alpha : I \to \mathbb{R}$ is a strictly increasing continuous function and $f : I \to \mathbb{R}$ is continuous on $[a, b]$ and α-differentiable on (a, b). Then there exists $c \in (a, b)$ such that*

$$D_\alpha f(c) = \frac{f(b) - f(a)}{\alpha(b) - \alpha(a)}$$

Proof. Let $g = \alpha$ in the previous theorem. □

The next result is a proof of L'Hopital's rule in the "0/0" case, based on the usual proof for the ordinary derivative.

Theorem 22. *(L'Hopital) Suppose $\alpha : (a, b) \to \mathbb{R}$ is a strictly increasing continuous function and $f, g : (a, b) \to \mathbb{R}$ are α-differentiable on (a, b). Suppose that*

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} g(x) = 0$$

If $D_\alpha g \neq 0$ on (a, b) and the limit

$$\lim_{x \to a^+} \frac{D_\alpha f(x)}{D_\alpha g(x)} = L$$

exists then $g \neq 0$ on (a, b) and

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$$

Proof. We extend f, g to functions on $[a, b]$ by defining $f(a) = f(b) = 0$. Hence, f and g are continuous on the interval $[a, x]$ for all $x < b$. We apply the Mean Value Theorem: if $x \in (a, b)$ then

$$g(x) = g(x) - g(a) = [D_\alpha g(c)](x - a) \quad \text{for some} \quad c \in (a, x)$$

We have $D_\alpha g(c) \neq 0$ and so $g(x) \neq 0$. By Cauchy's Mean Value Theorem, if $x \in (a, b)$ then there exists $c \in (a, x)$ such that

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{D_\alpha f(c)}{D_\alpha g(c)}$$

Since $c \in (a, x)$, we have $c \to a^+$ as $x \to a^+$. So

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{c \to a^+} \frac{D_\alpha f(c)}{D_\alpha g(c)} = L$$

□

We now see a result relating the sign of the α-derivative to the sign of the function, which reflects the corresponding theorem about the ordinary derivative.

Theorem 23. *Suppose $\alpha : I \to \mathbb{R}$ is a strictly increasing continuous function, and $f : I \to \mathbb{R}$ is α-differentiable on I. Then*

1. *If $D_\alpha f(x) \geq 0$ on I then f is increasing on I.*

2. *If $D_\alpha f(x) \leq 0$ on I then f is decreasing on I.*

3. *If $D_\alpha f(x) = 0$ on I then f is constant on I.*

Proof. Suppose that $x_1, x_2 \in I$ with $x_1 < x_2$. Since f is α-differentiable, we have

$$f(x_2) - f(x_1) = [\alpha(x_2) - \alpha(x_1)] D_\alpha f(x_0)$$

Since α is strictly increasing, then $\text{sign}(f(x_2) - f(x_1)) = \text{sign}[D_\alpha f(x_0)]$. □

We now apply the α-derivative to the concept of Taylor expansions to obtain the following result.

Theorem 24. *(Taylor Expansions) Let $n \in \mathbb{N}$, and let $f : I \to \mathbb{R}$ be a function such that $f, D_\alpha f, \ldots, D_\alpha^{(n)} f$ are continuous on I and $D_\alpha^{(n+1)} f$ exists on (a, b). If $x_0 \in I$ then for any $x \in I$ there exists c between x and x_0 such that*

$$f(x) = f(x_0) + D_\alpha f(x_0)(x - x_0) + \cdots + \frac{D_\alpha^{(n)} f(x_0)}{n!} + \frac{D_\alpha^{(n+1)} f(c)(x - x_0)^{n+1}}{(n+1)!}$$

Proof. Fix $x, x_0 \in I$. Denote by J the closed interval with endpoints x and x_0. We define a function $f : J \to \mathbb{R}$ by

$$F(t) := f(x) - f(t) - (x - t)D_\alpha f(t) - \cdots - \frac{(x - t)^n}{n!} D_\alpha^{(n)} f(t)$$

Then we have

$$D_\alpha F(t) = -\frac{(x - t)^n}{n!} D_\alpha^{(n+1)} f(t)$$

Now we define a function $G : J \to \mathbb{R}$ by

$$G(t) := F(t) - \left(\frac{x - t}{x - x_0}\right)^{n+1} F(x_0)$$

So we have $G(x_0) = G(x) = 0$ and can therefore apply Rolle's Theorem to conclude that there exists $c \in J$ such that

$$0 = D_\alpha G(c) = D_\alpha F(c) + (n + 1)\frac{(x - c)^n}{(x - x_0)^{n+1}} F(x_0)$$

Hence,

$$\begin{aligned} F(x_0) &= -\frac{1}{n+1} \frac{(x - x_0)^{n+1}}{(x - c)^n} F'(c) \\ &= \frac{1}{n+1} \frac{(x - x_0)^{n+1}}{(x - c)^n} \frac{(x - c)^n}{n!} D_\alpha^{(n+1)} f(c) \\ &= \frac{D_\alpha^{(n+1)} f(c)}{(n+1)!}(x - x_0)^{n+1} \end{aligned}$$

□

Chapter 5

The Fundamental Theorem

The Fundamental Theorem of Calculus has analogues in the context of both the Generalised Riemann and Riemann-Stieltjes integrals, but so far has not been proved for the Generalised Riemann-Stieltjes integral. The following result is made possible by the α-derivative in the previous chapter, as in the proof in the Riemann-Stieltjes case. However, it is also somewhat stronger than the usual Fundamental Theorem for the Riemann integral, since the integrability of f is a consequence of the theorem rather than an assumption. This is due to the method of proof used in the Fundamental Theorem for the Generalised Riemann integral.

Theorem 25. *(The Fundamental Theorem) Suppose $\alpha : I \to \mathbb{R}$ is a function which is continuous and strictly increasing. Suppose $f, F : I \to \mathbb{R}$ are functions such that*

1. *F is continuous.*
2. *$D_\alpha F(x) = f(x)$ for all $x \in I$.*

Then $f \in GRS(I, \alpha)$ and $\int_a^b f d\alpha = F(b) - F(a)$.

Proof. We want to construct a gauge $\delta_\varepsilon : I \to \mathbb{R}$. If $t \in I$, then the α-derivative of F at t exists and so there exists a constant $\delta_\varepsilon(t) > 0$ such that

$$0 < |z - t| < \delta_\varepsilon(t) \Rightarrow \left|\frac{F(z) - F(t)}{\alpha(z) - \alpha(t)} - f(t)\right| < \frac{\varepsilon}{2[\alpha(b) - \alpha(a)]}$$

Multiply by $|\alpha(z) - \alpha(t)|$ to obtain

$$|F(z) - F(t) - f(t)[\alpha(z) - \alpha(t)]| < \frac{\varepsilon|\alpha(z) - \alpha(t)|}{2[\alpha(b) - \alpha(a)]}$$

for $z \in [t - \delta_\varepsilon(t), t + \delta_\varepsilon(t)] \cap I$. We will choose δ_ε as our gauge. Now let $u, v \in I$, with $u < v$, such that $t \in [u, v] \subseteq [t - \delta_\varepsilon(t), t + \delta_\varepsilon(t)]$. We have, noting that α is increasing,

$$\begin{aligned}
|F(v) - F(u) - f(t)[\alpha(v) - \alpha(u)]| &\leq |F(v) - F(t) - f(t)[\alpha(v) - \alpha(t)]| \\
&\quad + |F(t) - F(u) - f(t)[\alpha(t) - \alpha(u)]| \\
&< \frac{1}{2[\alpha(b) - \alpha(a)]}\varepsilon[\alpha(v) - \alpha(t)] \\
&\quad + \frac{1}{2[\alpha(b) - \alpha(b)]}\varepsilon[\alpha(t) - \alpha(u)] \\
&= \frac{\varepsilon}{2[\alpha(b) - \alpha(a)]}[\alpha(v) - \alpha(u)]
\end{aligned}$$

We have the telescoping sum

$$F(b) - F(a) = \sum_{i=1}^{n}[F(x_i) - F(x_{i-1})]$$

If P is a δ_ε-fine partition then $t_i \in [x_{i-1}, x_i] \subseteq [t_i - \delta_\varepsilon(t_i), t_i + \delta_\varepsilon(t_i)]$. So

$$\begin{aligned}|F(b)-F(a)-S(f,P,\alpha)| &= \Big|\sum_{i=1}^{n}[F(x_i)-F(x_{i-1})]-f(t_i)[\alpha(x_i)-\alpha(x_{i-1})]\Big| \\ &\le \sum_{i=1}^{n}|F(x_i)-F(x_{i-1})-f(t_i)[\alpha(x_i)-\alpha(x_{i-1})]| \\ &< \sum_{i=1}^{n}\frac{\varepsilon}{2[\alpha(b)-\alpha(a)]}[\alpha(x_i)-\alpha(x_{i-1})] \\ &= \frac{\varepsilon}{2[\alpha(b)-\alpha(a)]}[\alpha(x_n)-\alpha(x_0)] \\ &= \frac{\varepsilon}{2[\alpha(b)-\alpha(a)]}[\alpha(b)-\alpha(a)]\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $f \in GRS(I,\alpha)$ and

$$\int_a^b f d\alpha = F(b) - F(a)$$

□

In the case of the Riemann-Stieltjes integral, we have the following theorem, a highly useful result which relates the integrand and integrator functions and gives a formula for integration by parts.

Theorem 26. *If $f \in R(I,\alpha)$ then $\alpha \in R(I,f)$ and*

$$\int_a^b f d\alpha + \int_a^b \alpha df = f(b)\alpha(b) - f(a)\alpha(a)$$

For the Generalised Riemann-Stieltjes integral, however, this result is not true. Consider the following counterexample:

Example 15. Define $f, \alpha : [0,1] \to \mathbb{R}$ by

$$f(x) = \alpha(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ 1 & \text{if} \quad x \in (0,1] \end{cases}$$

Now from example 8, we have $f \in GRS(I,\alpha)$ and $\alpha \in GRS(I,f)$, however

$$\int_a^b f d\alpha = \int_a^b \alpha df = 0$$

and

$$f(b)\alpha(b) - f(a)\alpha(a) = f(1)\alpha(1) - f(0)\alpha(0) = 1 \neq 0$$

So the formula does not hold. The problem is that the Generalised Riemann-Stieltjes integral can integrate a wider range of functions than the Riemann-Stieltjes can, but some of these functions are not as 'nice', and so this property is no longer true. Since $f \in RS(I, \alpha)$ implies $f \in GRS(I, \alpha)$ and the integrals are equal, we can however say that the above theorem is true for $f \in RS(I, \alpha)$ where the integrals in the formula are Generalised Riemann-Stieltjes integrals.

There is a form of integration by parts which works for Generalised Riemann-Stieltjes integrals. This result is based on a similar theorem concerning the Generalised Riemann integral, however, as in the Fundamental Theorem, we now need to consider α-derivatives instead of ordinary ones.

Theorem 27. *Let $\alpha : I \to \mathbb{R}$ be a continuous, strictly increasing function. Suppose that $F, G : I \to \mathbb{R}$ are α-differentiable on I. Then*

$$(D_\alpha F)G \in GRS(I, \alpha) \Leftrightarrow F(D_a G) \in GRS(I, \alpha)$$

We then have

$$\int_a^b (D_\alpha F) G d\alpha = (FG)|_a^b - \int_a^b F(D_\alpha G) d\alpha$$

Proof. Since the functions F, G are α-differentiable on I, then the product FG is also α-differentiable on I, and

$$D_\alpha(FG) = (D_\alpha F)G + F(D_\alpha G)$$

Note that since F, G are α-differentiable, they are also continuous, and hence their product FG is continuous. We can now apply the Fundamental Theorem to show that $D_\alpha(FG) \in GRS(I, \alpha)$ and

$$\int_a^b D_\alpha(FG) d\alpha = (FG)|_a^b$$

Now, since $D_\alpha(FG) = (D_\alpha F)G + F(D_\alpha G)$, we have

$$(D_\alpha F)G = D_\alpha(FG) - F(D_\alpha G)$$

and so $(D_\alpha F)G$ is integrable if and only if $F(D_\alpha G)$ is integrable. If so, then

$$\begin{aligned}\int_a^b (D_\alpha F)G d\alpha &= \int_a^b D_\alpha(FG) d\alpha - \int_a^b F(D_\alpha G) d\alpha \\ &= (FG)|_a^b - \int_a^b F(D_\alpha G) d\alpha\end{aligned}$$

□

As a consequence of the Fundamental Theorem, we have the following useful result which relates an integral on $[a, b]$ to an integral on $[\Phi(a), \Phi(b)]$ for a function Φ.

Theorem 28. *(Substitution Theorem) Let $I = [a, b]$ and $J = [c, d]$. Suppose that $\alpha : I \cup J \to \mathbb{R}$ is a continuous, strictly increasing function. If*

1. *The function $F : J \to \mathbb{R}$ is continuous and $D_\alpha F(x) = f(x)$ for all $x \in J$.*
2. *The function $\Phi : I \to \mathbb{R}$ is continuous and $D_\alpha \Phi(x) = \phi(x)$ for all $x \in I$.*
3. *We have $\Phi(I) \subset J$.*

Then $(f \circ \Phi) \cdot \phi \in GRS(I, \alpha)$ and $f \in GRS(\Phi(I), \alpha)$. Furthermore,

$$\int_a^b (f \circ \Phi) \cdot \phi d\alpha = (F \circ \Phi)|_a^b = \int_{\Phi(a)}^{\Phi(b)} f d\alpha$$

Proof. Since F and Φ are continuous, the composition $F \circ \Phi$ is continuous on I. We apply the Chain Rule:

$$D_\alpha(F \circ \Phi)(x) = D_\alpha F[\Phi(x)] D_\alpha \Phi(x) = (f \circ \Phi)(x)\phi(x)$$

for all $x \in I$. Using the Fundamental Theorem, we have $(f \circ \Phi) \cdot \phi \in GRS(I, \alpha)$ and

$$\int_a^b (f \circ \Phi) \cdot \phi d\alpha = (F \circ \Phi)_a^b = F[\Phi(b)] - F[\Phi(a)]$$

Now, we apply the Fundamental Theorem to F alone to conclude that $f \in GRS(J, \alpha)$. But $\Phi(I)$is a compact subinterval in J, and so f is integrable on $\Phi(I)$, and also on the compact subinterval with endpoints $\Phi(a)$ and $\Phi(b)$. If $\Phi(a) \leq \Phi(b)$ then, by the Fundamental Theorem, we have

$$\int_{\Phi(a)}^{\Phi(b)} f d\alpha = F|_{\Phi(a)}^{\Phi(b)} = F[\Phi(b)] - F[\Phi(a)]$$

If $\Phi(b) < \Phi(a)$ then we apply the Fundamental Theorem to the interval $[\Phi(b), \Phi(a)]$ to obtain

$$\int_{\Phi(a)}^{\Phi(b)} f d\alpha = -\int_{\Phi(b)}^{\Phi(a)} f d\alpha = -F|_{\Phi(b)}^{\Phi(a)} = F[\Phi(b)] - F[\Phi(a)]$$

So, in conclusion, we have

$$\int_a^b (f \circ \Phi) \cdot \phi d\alpha = (F \circ \Phi)|_a^b = \int_{\Phi(a)}^{\Phi(b)} f d\alpha$$

□

Chapter 6

Convergence and Absolute Integrability

In this chapter, we aim to prove analogues of important convergence results such as the Monotone and Dominated Convergence Theorems, familiar mainly from Lebesgue integration. These results are helpful because they allow the interchange of a limit with an integral, which is not usually possible and therefore a very powerful tool, in particular in calculation.

To start, we shall prove a useful lemma concerning partitions and their subsets in relation to integration.

Definition 17. A **subpartition** of I is a collection $\{J_j\}_{j=1}^{s}$ of non-overlapping closed intervals in I.

Remark. Any subset of a partition is a subpartition.

Definition 18. A **tagged subpartition** of I is a collection $P = \{(J_j, t_j)\}_{j=1}^{s}$ of ordered pairs where $\{J_j\}_{j=1}^{s}$ are intervals forming a subpartition of I and $t_j \in J_j$ for $j \in \{1, \dots, s\}$. The t_j are called **tags**.

We also need to make an appropriate modification to the notion of δ-fineness.

Definition 19. If δ is a gauge on I then the tagged subpartition P is δ-fine if

$$J_j \subseteq [t_j - \delta(t_j), t_j + \delta(t_j)] \quad \text{for all} \quad j \in \{1, \ldots, s\}$$

We shall also clarify our notation:

Definition 20. If $P = \{(J_j, t_j)\}_{j=1}^{s}$ is a tagged subpartition of I, then let

$$U(P) = \bigcup_{j=1}^{s} J_j$$

If $f \in GRS(I, \alpha)$, and we write $J_j = [x_{j-1}, x_j]$ then we define

$$S(f, P, \alpha) = \sum_{j=1}^{s} f(t_j)[\alpha(x_j) - \alpha(x_{j-1}]$$

and

$$\int_{U(P)} f d\alpha = \sum_{j=1}^{s} \int_{J_j} f d\alpha$$

Now we are ready to prove an analogue of the Saks-Henstock Lemma, a result which is important in the proofs of convergence theorems for the Generalised Riemann integral.

Theorem 29. *(Saks-Henstock Lemma for the Generalised Riemann-Stieltjes Integral) Let $\alpha : I \to \mathbb{R}$ be an integrator function. Let $f \in GRS(I, \alpha)$. For $\varepsilon > 0$ let $\delta_\varepsilon : I \to (0, \infty)$ be a gauge on I such that if P is a δ_ε-fine partition then $|S(f, P, \alpha) - \int_a^b f d\alpha| < \varepsilon$. If $P_0 = \{(J_j, t_j)\}_{j=1}^{s}$ is any δ_ε-fine subpartition of I then*

$$\left| \sum_{j=1}^{s} \left\{ f(t_j)[\alpha(x_j) - \alpha(x_{j-1})] - \int_{J_j} f d\alpha \right\} \right| = |S(f, P_0, \alpha) - \int_{U(P_0)} f d\alpha| < \varepsilon$$

Proof. Let $K_1, \ldots, K_m$ be closed subintervals in I such that $\{J_j\} \cup \{K_k\}$ forms a partition of I. Fix $\beta > 0$. Now, using Corollary 4, we know that f_{K_k} is integrable for $k \in \{1, \ldots, m\}$. So there exists a gauge $\delta_{\beta,k}$ on K_k such that if Q_k is a $\delta_{\beta,k}$-fine partition of K_k then

$$|S(f, Q_k, \alpha) - \int_{K_k} f d\alpha| < \frac{\beta}{m}$$

We assume that $\delta_{\beta,k}(x) \leq \delta_\varepsilon(x)$ for all $x \in K_k$, as otherwise we could just set $\delta_{\beta,k}(x) = \delta_\varepsilon(x)$. Let P^* be the tagged partition $P^* := P_0 \cup Q_1 \cup \cdots \cup Q_m$ of I. We know that P^* is a δ_ε-fine partition since each element of the union is. So

$$|S(f, P^*, \alpha) - \int_I f d\alpha| < \varepsilon$$

Now, we have

$$S(f, P^*, \alpha) = S(f, P_0, \alpha) + S(f, Q_1, \alpha) + \cdots S(f, Q_m, \alpha)$$

and

$$\int_I f d\alpha = \int_{U(P_0)} f d\alpha + \int_{K_1} f d\alpha + \cdots + \int_{K_m} f d\alpha$$

So, by rearranging, we obtain

$$\begin{aligned}
&|S(f, P_0, \alpha) - \int_{U(P_0)} f d\alpha| \\
=&\big|[S(f, P^*, \alpha) - \sum_{k=1}^{m} S(f, Q_k, \alpha)] - [\int_I f d\alpha - \sum_{k=1}^{m} \int_{K_k} f d\alpha]\big| \\
=&\big|[S(f, P^*, \alpha) - \int_I f d\alpha] - [\sum_{k=1}^{m} S(f, Q_k, \alpha) - \sum_{k=1}^{m} \int_{K_k} f d\alpha]\big| \\
\leq&\big|S(f, P^*, \alpha) - \int_I f d\alpha| + |\sum_{k=1}^{m} S(f, Q_k, \alpha) - \sum_{k=1}^{m} \int_{K_k} f d\alpha\big| \\
<&\varepsilon + m(\frac{\beta}{m}) \\
=&\varepsilon + \beta
\end{aligned} \tag{6.1}$$

Since $\beta > 0$ was arbitrary, we conclude that $|S(f, P_0, \alpha) - \int_{U(P_0)} f d\alpha| < \varepsilon$. $\square$

Corollary 5. *Under the conditions of Theorem 29, we have*

$$\sum_{j=1}^{s} |f(t_j)[\alpha(x_j) - \alpha(x_{j-1})] - \int_{J_j} f d\alpha| < 2\varepsilon$$

Proof. Let P_0^+ be the set of pairs in P_0 for which $f(t_j)[\alpha(x_j) - \alpha(x_{j-1})] - \int_{J_j} f d\alpha \geq 0$. Let P_0^- be the set of pairs in P_0 for which $f(t_j)[\alpha(x_j) - \alpha(x_{j-1})] - \int_{J_j} f d\alpha < 0$. We apply Theorem 29 to each of these sets separately to obtain

$$\sum_{P_0^+} |f(t_j)[\alpha(x_j) - \alpha(x_{j-1})] - \int_{J_j} f d\alpha| = \sum_{P_0^+} f(t_j)[\alpha(x_j) - \alpha(x_{j-1})] - \int_{J_j} f d\alpha < \varepsilon$$

and

$$\begin{aligned} &\sum_{P_0^-} |f(t_j)[\alpha(x_j) - \alpha(x_{j-1})] - \int_{J_j} f d\alpha| \\ =& - \big(\sum_{P_0^-} f(t_j)[\alpha(x_j) - \alpha(x_{j-1})] - \int_{J_j} f d\alpha \big) \\ <& \varepsilon \end{aligned} \tag{6.2}$$

Adding these two inequalities, we obtain

$$\begin{aligned} &\sum_{j=1}^{s} |f(t_j)[\alpha(x_j) - \alpha(x_{j-1})] - \int_{J_j} f d\alpha| \\ =& \sum_{P_0^+} f(t_j)[\alpha(x_j) - \alpha(x_{j-1})] - \int_{J_j} f d\alpha \\ &- \sum_{P_0^-} f(t_j)[\alpha(x_j) - \alpha(x_{j-1})] - \int_{J_j} f d\alpha \\ =& \big(\sum_{P_0^+} f(t_j)[\alpha(x_j) - \alpha(x_{j-1})] - \int_{J_j} f d\alpha \big) \\ &- \big(\sum_{P_0^-} f(t_j)[\alpha(x_j) - \alpha(x_{j-1})] - \int_{J_j} f d\alpha \big) \\ <& 2\varepsilon \end{aligned} \tag{6.3}$$

$\square$

Corollary 6. *Under the conditions of theorem 29, we have*

$$\left|\sum_{j=1}^{s} |f(t_j)||\alpha(x_j) - \alpha(x_{j-1})| - \sum_{j=1}^{s}\left|\int_{J_j} f d\alpha\right|\right| < 2\varepsilon$$

Proof. One version of the Triangle Inequality states that for $A, B \in \mathbb{R}$,

$$-|A - B| \leq |A| - |B| \leq |A - B|$$

Let $A := f(t_j)[\alpha(x_j) - \alpha(x_{j-1})]$ and $B := \int_{J_j} f d\alpha$. Substituting into the above formula, we have

$$\begin{aligned} -|f(t_j)[\alpha(x_j) - \alpha(x_{j-1})] - \int_{J_j} f d\alpha| &\leq |f(t_j)[\alpha(x_j) - \alpha(x_{j-1})]| - \left|\int_{J_j} f d\alpha\right| \\ &\leq |f(t_j)[\alpha(x_j) - \alpha(x_{j-1})] - \int_{J_j} f d\alpha| \end{aligned}$$

Now sum from $j = 1$ to $j = s$ to obtain

$$\begin{aligned} &-\sum_{j=1}^{s} \left|f(t_j)[\alpha(x_j) - \alpha(x_{j-1})] - \int_{J_j} f d\alpha\right| \\ &\leq \sum_{j=1}^{s} \left|f(t_j)[\alpha(x_j) - \alpha(x_{j-1})]\right| - \left|\int_{J_j} f d\alpha\right| \\ &\leq \sum_{j=1}^{s} \left|f(t_j)[\alpha(x_j) - \alpha(x_{j-1})] - \int_{J_j} f d\alpha\right| \end{aligned}$$

Using the definition of the absolute value, we have

$$\begin{aligned} &\left|\sum_{j=1}^{s} |f(t_j)[\alpha(x_j) - \alpha(x_{j-1})]| - \left|\int_{J_j} f d\alpha\right|\right| \\ \leq &\sum_{j=1}^{s} \left|f(t_j)[\alpha(x_j) - \alpha(x_{j-1})] - \int_{J_j} f d\alpha\right| < 2\varepsilon \end{aligned}$$

using the previous corollary. □

Remark. Note that in the last corollary, if α is increasing then $|\alpha(x_j) - \alpha(x_{j-1})| = \alpha(x_j) - \alpha(x_{j-1})$.

We now examine the concept of absolute integrability, which further clarifies the relationship between integrals and absolute value.

Definition 21. A function $f : I \to \mathbb{R}$ with $f \in GRS(I, \alpha)$ is **absolutely integrable** with respect to α if $|f| \in GRS(I, \alpha)$.

Definition 22. Let $\phi : I \to \mathbb{R}$. The **variation** of ϕ over I is given by

$$Var(\phi, I) := \sup \left\{ \sum_{i=1}^{n} |\phi(x_i) - \phi(x_{i-1})| : P = \{I_i\} \quad \text{is a partition of I} \right\}$$

We say that ϕ has **bounded variation** on I if $Var(\phi, I) < \infty$.

Theorem 30. *(Characterisation of Absolute Integrability) Let $\alpha : I \to \mathbb{R}$ be an increasing function. Let $f \in GRS(I, \alpha)$. Then $|f|$ is integrable if and only if the indefinite integral $F(x) := \int_a^x f d\alpha$ has bounded variation on I for all $x \in I$. Then we have*

$$\int_I |f| d\alpha = Var(F, I)$$

Proof. For ($\Rightarrow$): If $|f| \in GRS(I, \alpha)$ and $Q := \{e_0, e_1, \ldots, e_m\}$ is any partition of I, then we have

$$\begin{aligned} \sum_{i=1}^{m} |F(e_i) - F(e_{i-1})| &= \sum_{i=1}^{m} \left| \int_{e_{i-1}}^{e_i} f d\alpha \right| \\ &\leq \sum_{i=1}^{m} \int_{e_{i-1}}^{e_i} |f| d\alpha \\ &= \int_a^b |f| d\alpha \end{aligned}$$

So F is of bounded variation on I since

$$Var(F, I) \leq \int_I |f| d\alpha$$

For ($\Leftarrow$): Suppose F is of bounded variation on I, and that $Var(F, I) < \infty$. Fix $\varepsilon > 0$, and let $Q := \{e_0, e_1, \ldots, e_m\}$ be a partition of I such that

$$Var(F, I) - \varepsilon \leq \sum_{i=1}^{m} |F(e_i) - F(e_{i-1})| \leq Var(F, I)$$

If $e^* \in (e_{i-1}, e_i)$ then we have

$$|F(e_i) - F(e_{i-1})| \leq |F(e^*) - F(e_{i-1})| + |F(e_i) - F(e^*)|$$

So by induction, we can add a finite number of additional points to the partition Q, and the summation above will not exceed $Var(F, I)$ as it is the supremum (although the sum will increase). Let δ_ε be a gauge such that for any δ_ε-fine partition P of I, we have

$$\left| S(f, P, \alpha) - \int_a^b f d\alpha \right| < \frac{\varepsilon}{3}$$

Now, using Corollary 6 of the Saks-Henstock Lemma, we conclude that

$$\left| \sum_{i=1}^{n} |f(t_i)|[\alpha(x_i) - \alpha(x_{i-1})] - \sum_{i=1}^{n} \left| \int_{I_i} f d\alpha \right| \right| \leq 2\frac{\varepsilon}{3} \tag{6.4}$$

We define the set $E := \{e_i : i = 0, \ldots, m\}$, and define a gauge δ_ε^* on I by

$$\delta_\varepsilon^*(t) := \min\{\delta_\varepsilon(t), \frac{1}{2} dist(t, E - \{t\})\}$$

Now if P^* is δ_ε^*-fine then it is also δ_ε-fine, and so (6.4) holds for P^*. From the definition of δ_ε^*, every $e_j \in E$ is the tag of at least one subinterval in P^*. We will add a finite number of points to the set of partition points of P^*, namely those e_j which are not already partition points. Let $\{u_i\}$ denote the set of partition points of P^* after performing this procedure, and let $\{\tau_i\}$ denote the corresponding tags. We have intervals $J_i := [u_{i-1}, u_i]$. So we have

$$Var(F, I) - \frac{\varepsilon}{3} \leq \sum_{i=1}^{p} |F(u_i) - F(u_{i-1})| = \sum_{i=1}^{p} \left| \int_{J_j} f d\alpha \right| \leq Var(F, I)$$

Combining with (6.4) above, for a partition P which is δ_ε^*-fine we obtain

$$\begin{aligned}|S(|f|,P,\alpha)-Var(F,I)| &\leq \Big|\sum_{i=1}^{p}|f(\tau_j)|[\alpha(u_j)-\alpha(u_{j-1})] \\ &\quad -\sum_{i=1}^{p}\Big|\int_{J_j} fd\alpha\Big|\Big| + \Big|\sum_{i=1}^{p}\Big|\int_{J_j} fd\alpha\Big| - Var(F,I)\Big| \\ &\leq \frac{2\varepsilon}{3}+\frac{\varepsilon}{3} \\ &= \varepsilon\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $|f| \in GRS(I,\alpha)$ with integral Var(F,I). ☐

Theorem 31. *(Comparison Test) Let $\alpha : I \to \mathbb{R}$ be an increasing function. If $f, g \in GRS(I,\alpha)$ and $|f(x)| \leq g(x)$ for $x \in I$ then f is absolutely integrable and*

$$\Big|\int_I fd\alpha \leq \int_I |f|d\alpha \leq \int_I gd\alpha$$

Proof. We define $F(x) := \int_a^x fd\alpha$, so that if P is a partition of I then

$$\begin{aligned}|F(x_i)-F(x_{i-1})| &= \left|\int_a^{x_i} fd\alpha - \int_a^{x_{i-1}} fd\alpha\right| \\ &= \left|\int_a^{x_i} fd\alpha + \int_{x_{i-1}}^{a} fd\alpha\right| \\ &= \left|\int_{x_{i-1}}^{x_i} fd\alpha\right|\end{aligned}$$

Now $f(x) \leq |f(x)| \leq g(x)$ for $x \in I$, and so

$$f \leq g \quad \text{on} \quad I \Rightarrow \int_I fd\alpha \leq \int_I gd\alpha$$

Also, $-f(x) \leq |f(x)| \leq g(x)$ for $x \in I$ and so $f(x) \geq -g(x)$ for $x \in I$. Hence

$$-g \leq f \quad \text{on} \quad I \Rightarrow \int_I (-g)d\alpha \leq \int_I fd\alpha$$

So we have

$$-\int_I gd\alpha \leq \int_I fd\alpha \leq \int_I gd\alpha$$

or equivalently,

$$\left|\int_I f d\alpha\right| \leq \int_I g d\alpha$$

Hence

$$\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| \leq \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} g d\alpha = \int_a^b g d\alpha$$

and so we conclude that $Var(F, I) \leq \int_a^b g d\alpha$. By the Characterisation of Absolute Integrability, we conclude that $|f| \in GRS(I, \alpha)$ and, by Corollary 61 and monotonicity,

$$\left|\int_I f d\alpha\right| \leq \int_I |f| d\alpha \leq \int_I g d\alpha$$

□

Theorem 32. *Let $\alpha : I \to \mathbb{R}$ be an increasing functions. If $f, g : I \to \mathbb{R}$ are absolutely integrable with respect to α on I, and $c \in \mathbb{R}$, then cf and $f + g$ are absolutely integrable.*

Remark. This theorem shows that the set of absolutely integrable functions on I forms a vector space.

Proof. We have $|cf|(x) = |c||f(x)|$, and since the product of a constant with an integrable function is integrable, we conclude that $|cf| \in GRS(I, \alpha)$. Since $|f|, |g| \in GRS(I, \alpha)$ then $|f| + |g| \in GRS(I, \alpha)$. We have $|f + g| \leq |f| + |g|$, so by the Comparison Test, we conclude that $|f + g| \in GRS(I, \alpha)$. □

Theorem 33. *If $f \in GRS(I, \alpha)$ then the following are equivalent:*

1. *$|f| \in GRS(I, \alpha)$.*

2. *There exists a function $w : I \to \mathbb{R}$ which is absolutely integrable and satisfies $f(x) \leq w(x)$ for all $x \in I$.*

3. *There exists a function $\beta : I \to \mathbb{R}$ which is absolutely integrable and satisfies $\beta(x) \le f(x)$ for all $x \in I$.*

Proof. For (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3): Let $w = f$.
For (2) $\Rightarrow$ (1): We have $f = w - (w - f)$. Since $w - f$ is integrable and $w - f \ge 0$ on I, we know that $w - f$ is absolutely integrable. By Theorem 32, we conclude that f is absolutely integrable.
For (3) $\Rightarrow$ (1): We have $f = \beta + (f - \beta)$. Since $f - \beta$ is integrable and $f - \beta \ge 0$ on I, we know that $f - \beta$ is absolutely integrable. By Theorem 32, we conclude that f is absolutely integrable. □

Corollary 7. *If $\alpha : I \to \mathbb{R}$ is an increasing function, and $f \in GRS(I, \alpha)$, then f is absolutely integrable if any of the following conditions hold:*

- *The function f is bounded above on I.*
- *The function f is bounded below on I.*
- *The function f is bounded on I.*

Proof. If f is bounded above on I then there exists $M \in \mathbb{R}$ such that $|f(x)| \le M$ for $x \in I$. Since constant functions are positive, they are absolutely integrable, and so we conclude by the previous theorem that f is absolutely integrable. The proofs for f bounded below, or f bounded, are similar. □

We introduce the following notation:

$$f \vee g = \max\{f, g\} \quad \text{and} \quad f \wedge g = \min\{f, g\}$$

$$f^+ = f \vee 0 \quad \text{and} \quad f^- = (-f) \vee 0$$

Theorem 34. *If $\alpha : I \to \mathbb{R}$ is an increasing function and $f \in GRS(I, \alpha)$ then the following are equivalent:*

1. *The function f is absolutely integrable.*

2. $f^+, f^- \in GRS(I, \alpha)$.

3. *The functions f^+, f^- are absolutely integrable.*

Proof. For (1) $\Rightarrow$ (2): We have $f^+ = \frac{1}{2}(f + |f|)$ and $f^- = \frac{1}{2}(|f| - f)$. Since $f, |f| \in GRS(I, \alpha)$, we have $f^*, f^- \in GRS(I, \alpha)$.
For (2) $\Rightarrow$ (3): Since $f^+, f^- \geq 0$, the functions f^+, f^- are absolutely integrable.
For (3) $\Rightarrow$ (1): We have $|f| = f^+ + f^-$. Since f^+, f^- are absolutely integrable, we conclude that $|f|$ is absolutely integrable. □

Theorem 35. *If $\alpha : I \to \mathbb{R}$ is an increasing function and $f, g \in GRS(I, \alpha)$ then the following are equivalent:*

1. *The functions f, g are absolutely integrable.*

2. *The function $f \vee g$ is absolutely integrable.*

3. *The function $f \wedge g$ is absolutely integrable.*

Proof. For (1) $\Rightarrow$ (2): We have $f \vee g = \frac{1}{2}(f + g + |f - g|)$, which is a linear combination of absolutely integrable functions, and so $f \vee g$ is absolutely integrable.
For (2) $\Rightarrow$ (1): Since $f, g \leq f \vee g$, by theorem 33 we conclude that f and g are absolutely integrable.
For (1) $\Rightarrow$ (3): We have $f \wedge g = \frac{1}{2}(f + g - |f - g|)$, which is a linear combination of absolutely integrable functions, and so $f \wedge g$ is absolutely integrable.
For (3) $\Rightarrow$ (1): Since $f, g \geq f \wedge f$, by theorem 33 we conclude that f and g are absolutely integrable. □

Theorem 36. *If $\alpha : I \to \mathbb{R}$ is an increasing function and $f, g, \beta, w \in GRS(I, \alpha)$ then*

1. *If $f \leq w$ and $g \leq w$ then $f \vee g$ and $f \wedge g$ are integrable.*
2. *If $\beta \leq f$ and $\beta \leq g$ then $f \vee g$ and $f \wedge g$ are integrable.*

Proof. Note that

$$f \vee g = \frac{1}{2}(f + g + |f - g|) \quad \text{and} \quad f \wedge g = \frac{1}{2}(f + g - |f - g|)$$

For (1): We have $f \vee g \leq w$, and sp

$$0 \leq |f - g| = 2(f \vee g) - f - g \leq 2w - f - g$$

Using the Comparison Test, we know that $|f - g|$ is integrable, and so $f \vee g$ and $f \wedge g$ are integrable.
For (2): We have $f \wedge g \geq \beta$ and therefore $-(f \wedge g) \leq \beta$. Now

$$\begin{aligned} f \wedge g &= \frac{1}{2}(f + g - |f - g|) \\ \Rightarrow & 2(f \wedge g) = f + g - |f - g| \\ \Rightarrow & |f - g| = f + g - 2(f \wedge g) \\ \Rightarrow & 0 \leq |f - g| \leq f + g - 2\beta \end{aligned}$$

By the Comparison Test, we know that $|f - g|$ is integrable, and so $f \vee g$ and $f \wedge g$ are integrable. □

Remark. Using mathematical induction, we can show that if $f_1, \ldots, f_n$ are integrable then $f_1 \vee \cdots \vee f_n$ and $f_1 \wedge \cdots \wedge f_n$ are integrable.

Definition 23. A sequence of functions (f_k), where $f_k : I \to \mathbb{R}$, is said to be **uniformly convergent** on I to a function f if for every $\varepsilon > 0$ there exists $K_\varepsilon \in \mathbb{N}$ such that if $k \geq K_\varepsilon$ and $x \in I$ then $|f_k(x) - f(x)| < \varepsilon$.

Theorem 37. *(Uniform Convergence Theorem) Let $\alpha : I \to \mathbb{R}$ be an increasing function. If the sequence $(f_k) \in GRS(I, \alpha)$ converges to f uniformly on I then $f \in GRS(I, \alpha)$ and*

$$\int_I f d\alpha = \lim_{k\to\infty} \int_I f_k d\alpha$$

Proof. Fix $\varepsilon > 0$. By uniform convergence, there exists $K_\varepsilon \in \mathbb{N}$ such that if $k \geq K_\varepsilon$ and $x \in I$ then $|f_k(x) - f(x)| < \frac{\varepsilon}{2[\alpha(b)-\alpha(a)]}$. So if $h, k \geq K_\varepsilon$ then

$$|f_k(x) - f(x)| < \frac{\varepsilon}{3} \quad \text{and} \quad |f_h(x) - f(x)| < \frac{\varepsilon}{2[\alpha(b) - \alpha(a)]}$$

Hence

$$\begin{aligned} |f_h(x) - f_k(x)| &= |[f_h(x) - f(x)] + [f(x) - f_k(x)]| \\ &\leq |f_h(x) - f(x)| + |f(x) - f_k(x)| \\ &< \frac{\varepsilon}{[\alpha(b) - \alpha(a)]} \end{aligned}$$

By expanding the absolute value, we have

$$-\frac{\varepsilon}{[\alpha(b) - \alpha(a)]} < f_h(x) - f_k(x) < \frac{\varepsilon}{[\alpha(b) - \alpha(a)]} \quad \text{for} \quad x \in I$$

So after integrating we obtain

$$\begin{aligned} -\varepsilon = -\frac{\varepsilon}{[\alpha(b) - \alpha(a)]}[\alpha(b) - \alpha(a)] &< \int_I f_h d\alpha - \int_I f_k d\alpha \\ &< \frac{\varepsilon}{[\alpha(b) - \alpha(a)]}[\alpha(b) - \alpha(a)] = \varepsilon \end{aligned} \tag{6.5}$$

and so

$$\left| \int_I f_h d\alpha - \int_I f_k d\alpha \right| < \varepsilon$$

Now $\varepsilon > 0$ is arbitrary, so we conclude that $(\int_I f_k)$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, it is also a convergent sequence in $\mathbb{R}$, with limit $A \in \mathbb{R}$. We will show that $f \in GRS(I, \alpha)$ with integral A. Fix $\varepsilon > 0$. Let $K_\varepsilon \in \mathbb{N}$

be as above, and suppose that P is a tagged partition of I. For $k \geq K_\varepsilon$ we have

$$\begin{aligned}
|S(f_k, P, \alpha) - S(f, P, \alpha)| &= |\sum_{i=1}^{n}[f_k(t_i) - f(t_i)][\alpha(x_i) - \alpha(x_{i-1})]| \\
&\leq \sum_{i=1}^{n} |f_k(t_i) - f(t_i)|[\alpha(x_i) - \alpha(x_{i-1})] \\
&< \sum_{i=1}^{n} \frac{\varepsilon}{[\alpha(b) - \alpha(a) + 2]}[\alpha(x_i) - \alpha(x_{i-1})] \\
&= \frac{\varepsilon}{[\alpha(b) - \alpha(a) + 2]}[\alpha(b) - \alpha(a)]
\end{aligned}$$

The second last line is true by uniform convergence, while the last line is derived by evaluating the telescoping sum. Let $r \geq K_\varepsilon$ be such that $|\int_I f_r d\alpha - A| < \varepsilon$. Since $f_r \in GRS(I, \alpha)$, there exists a gauge $\delta_{r,\varepsilon}$ on I such that

$$\left|\int_I f_r d\alpha - S(f_r, P, \alpha)\right| < \frac{\varepsilon}{[\alpha(b) - \alpha(a) + 2]}$$

whenever P is a $\delta_{r,\varepsilon}$-fine partition. We then have

$$\begin{aligned}
|S(f, P, \alpha) - A| &\leq |S(f, P, \alpha) - S(f_r, P, \alpha)| + |S(f_r, P, \alpha) - \int_I f_r d\alpha| \\
&+ |\int_I f_r d\alpha - A| \\
&< \frac{\varepsilon}{[\alpha(b) - \alpha(a) + 2]}[\alpha(b) - \alpha(a)] + \frac{\varepsilon}{[\alpha(b) - \alpha(a) + 2]} \\
&+ \frac{\varepsilon}{[\alpha(b) - \alpha(a) + 2]} \\
&= \frac{\varepsilon}{[\alpha(b) - \alpha(a) + 2]}[\alpha(b) - \alpha(a) + 2] \\
&= \varepsilon
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $f \in GRS(I, \alpha)$ and $\int_I f d\alpha = A$. □

Theorem 38. *(Monotone Convergence Theorem) Let $\alpha : I \to \mathbb{R}$ be an increasing function. Let $(f_k)_{k=1}^{\infty}$ be a monotone sequence in $GRS(I, \alpha)$. Let*

$f(x) := \lim_{k\to\infty} f_k(x)$ for all $x \in I$. Then $f \in GRS(I,\alpha)$ if and only if the sequence $(\int_I f_k d\alpha)_{k=1}^{\infty}$ is bounded in $\mathbb{R}$. Furthermore,

$$\int_I f d\alpha = \lim_{k\to\infty} \int_I f_k d\alpha$$

Proof. Without loss of generality, we may assume that the sequence $(f_k)_{k=1}^{\infty}$ is monotone increasing.

For ($\Rightarrow$): If $f \in GRS(I,\alpha)$ then $f_1(x) \le f_k(x) \le f_{k+1}(x) \le f(x)$ for all $x \in I$, and therefore, since α is increasing, we have

$$\int_I f_1 d\alpha \le \int_I f_k d\alpha \le \int_I f_{k+1} d\alpha \le \int_I f d\alpha$$

So the sequence $(\int_I f_k d\alpha)_{k=1}^{\infty}$ is increasing and bounded. Since we are proving an if and only if statement, it suffices to show that the integral of the limit is the limit of the integral in only one direction (here we will use the ($\Leftarrow$) direction).

For ($\Leftarrow$): Let $A := \sup\{\int_I f_k d\alpha : k \in \mathbb{N}\}$. The increasing, bounded sequence $(\int_I f_k d\alpha)$ converges to its supremum, that is, A. If $\varepsilon > 0$, let $r \in \mathbb{N}$ be such that

$$\frac{1}{2^{r-2}} < \frac{\varepsilon}{[\alpha(b)-\alpha(a)+2]} \quad \text{and} \quad 0 \le A - \int_I f_r < \frac{\varepsilon}{[\alpha(b)-\alpha(a)+2]}$$

Since $f_k \in GRS(I,\alpha)$ for all $k \in \mathbb{N}$, there exists a gauge $\delta_k : I \to (0,\infty)$ such that if P is a δ_k-fine partition then

$$|S(f_k, P, \alpha) - \int_I f_k d\alpha| < \frac{1}{2^k}$$

Now since $f(x) = \lim_{k\to\infty} f_k(x)$, for all $x \in I$ there exists an integer $k(x) \ge r$ such that

$$0 \le f(x) - f_{k(x)}(x) < \frac{\varepsilon}{[\alpha(b)-\alpha(a)+2]}$$

We define $\delta_\varepsilon(t) := \delta_{k(t)}(t)$ for $t \in I$, so δ_ε is a gauge on I. We want to show that f is integrable with integral A, that is, if P is a δ_ε-fine partition then we want to show that $|S(f,P,\alpha) - A|$ is suitably small. Using the Triangle Inequality, we have

$$\begin{aligned}|S(f,P,\alpha) - A| \leq & |\sum_{i=1}^{n} f(t_i)[\alpha(x_i) - \alpha(x_{i-1})] - \sum_{i=1}^{n} f_{k(t_i)}(t_i)[\alpha(x_i) - \alpha(x_{i-1})]| \\ & + |\sum_{i=1}^{n} f_{k(t_i)}(t_i)[\alpha(x_i) - \alpha(x_{i-1})] - \sum_{i=1}^{n} \int_{I_i} f_{k(t_i)} d\alpha| \\ & + |\sum_{i=1}^{n} \int_{I_i} f_{k(t_i)} d\alpha - A|\end{aligned} \tag{6.6}$$

We shall look at each term separately. For the first term, we have

$$\begin{aligned}& |\sum_{i=1}^{n} f(t_i)(\alpha(x_i) - \alpha(x_{i-1})) - \sum_{i=1}^{n} f_{k(t_i)}(t_i)[\alpha(x_i) - \alpha(x_{i-1})]| \\ \leq & \sum_{i=1}^{n} |f(t_i) - f_{k(t_i)}(t_i)|[\alpha(x_i) - \alpha(x_{i-1})] \\ \leq & \frac{\varepsilon}{[\alpha(b) - \alpha(a) + 2]} \sum_{i=1}^{n} [\alpha(x_i) - \alpha(x_{i-1})] \\ = & \frac{\varepsilon}{[\alpha(b) - \alpha(a) + 2]} [\alpha(b) - \alpha(a)]\end{aligned}$$

The second last line is true since $0 \leq f(x) - f_k(x) < \frac{\varepsilon}{[\alpha(b) - \alpha(a) + 2]}$ For the second term, we have

$$\begin{aligned}& |\sum_{i=1}^{n} f_{k(t_i)}(t_i)[\alpha(x_i) - \alpha(x_{i-1})] - \sum_{i=1}^{n} \int_{I_i} f_{k(t_i)} d\alpha| \\ \leq & \sum_{i=1}^{n} |f_{k(t_i)}(t_i)[\alpha(x_i) - \alpha(x_{i-1})] - \sum_{i=1}^{n} \int_{I_i} f_{k(t_i)} d\alpha|\end{aligned}$$

Let $s := \max\{k(t_1), \ldots, k(t_n)\}$. So we must have $s \geq r$. Now, we rewrite the summation above by iterating: firstly over the integers i such that $k(t_i) = p$ for some integer $p \geq r$, and secondly over $p = r, \ldots, s$. Consider the tags t_i such that $k(t_i) = p$ for some fixed value of p. The interval I_i (which contains t_i) must then be contained within a closed interval centred at t_i and with radius $\delta_\varepsilon(t_i) = \delta_{k(t_i)}(t_i) = \delta_p(t_i)$. Hence, the collection $P_0 = \{(I_i, t_i) : k(t_i) = p\}$ forms a δ_p-fine subpartition. Due to δ_p-fineness, we have

$$|S(f_p, P_0, \alpha) - \int_I f_p d\alpha| \leq \frac{1}{2p}$$

Using Corollary 5 of the Saks-Henstock Lemma, we have

$$\sum_{k(t_i)=p} |f_{k(t_i)}(t_i)[\alpha(x_i) - \alpha(x_{i-1})] - \int_I f_{k(t_i)} d\alpha| \leq 2 \cdot \frac{1}{2p} = \frac{1}{2^{p-1}}$$

Now, taking a summation over $p = r, \ldots, s$ we conclude that the second term in (6.6) is dominated by

$$\sum_{p=r}^{s} \frac{1}{2^{p-1}} < \sum_{p=r}^{\infty} \frac{1}{2^{p-1}} = \frac{1}{2^{r-2}} < \frac{\varepsilon}{[\alpha(b) - \alpha(a) + 2]}$$

by the definition of r. For the third term: since (f_k) is increasing and $r \leq k(t_i) \leq s$, we have $f_r \leq f_{k(t_i)} \leq f_s$ and therefore

$$\int_{I_i} f_r d\alpha \leq \int_{I_i} f_{k(t_i)} d\alpha \leq \int_{I_i} f_s d\alpha$$

We take a summation from $i = 1$ to $i = n$ to obtain

$$\int_I f_r d\alpha \leq \sum_{i=1}^{n} \int_{I_i} f_{k(t_i)} d\alpha \leq \int_I f_r d\alpha$$

Hence, by the definitions of A and r, we have

$$A - \frac{\varepsilon}{[\alpha(b) - \alpha(a) + 2]} \leq \sum_{i=1}^{n} \int_{I_i} f_{k(t_i)} d\alpha \leq A$$

So we have

$$\sum_{i=1}^{n} \int_{I_i} f_{k(t_i)} d\alpha - A| < \frac{\varepsilon}{[\alpha(b) - \alpha(a) + 2]}$$

Thus, we have estimated each of the three terms in equation (6.6), and can now combine then. So if P is δ_ε-fine, then

$$\begin{aligned}|S(f,P,\alpha)-A| &\le (b-a)\frac{\varepsilon}{[\alpha(b)-\alpha(a)+2]}+\frac{\varepsilon}{[\alpha(b)-\alpha(a)+2]}\\ &+\frac{\varepsilon}{[\alpha(b)-\alpha(a)+2]}\\ &=(b-a+2)\frac{\varepsilon}{[\alpha(b)-\alpha(a)+2]}\\ &=\varepsilon\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $f \in GRS(I,\alpha)$ and

$$\int_I f d\alpha = \lim_{k\to} \int_I f_k d\alpha$$

□

Lemma 1. *Let $f_k, \beta \in GRS(I,\alpha)$ be such that*

$$\beta(x) \le f_k(x) \quad \textit{for} \quad x \in I, k \in \mathbb{N}$$

Then $\inf\{f_k : k \in \mathbb{N}\}$ *belongs to* $GRS(I,\alpha)$*.*

Proof. Since $\beta(x) \le f_k(x)$ for all $x \in I$ and all $k \in \mathbb{N}$, we know that $\inf\{f_k\}$ exists and is at least β. If $k \in \mathbb{N}$, then define the function $\psi_k = f_1 \wedge \cdots \wedge f_k$. Using induction, we conclude that $\psi_k \in GRS(I,\alpha)$. Now, the sequence $\{\psi_k\}$ is decreasing on I to the limit $\inf\{f_k\}$. But

$$\int_I \psi_k d\alpha \ge \int_I \beta d\alpha$$

So we can apply the Monotone Convergence Theorem to conclude that

$$\lim_{k\to\infty} \psi_k \in GRS(I,\alpha)$$

Hence $\inf\{f_k\} \in GRS(I,\alpha)$. □

Theorem 39. *(Fatou's Lemma) Let* $\alpha : I \to \mathbb{R}$ *be an increasing function. Let* $f_k, \beta \in GRS(I, \alpha)$ *be such that* $\beta(x) \leq f_k(x)$ *for all* $x \in I$ *and* $k \in \mathbb{N}$*, and*

$$\liminf_{k\to\infty} \int_I f_k d\alpha < \infty$$

Then $\liminf_{k\to\infty} f_k \in GRS(I, \alpha)$ *and*

$$-\infty < \int_I \liminf_{k\to\infty} f_k d\alpha \leq \liminf_{k\to\infty} \int_I f_k d\alpha < \infty$$

Proof. We define $\phi_k := \inf\{f_m : m \geq k, m \in \mathbb{N}\}$ for $k \in \mathbb{N}$. Now, from the previous lemma, we have $\phi_k \in GRS(I, \alpha)$. Since $\beta(x) \leq \phi_k(x) \leq f_k(x)$ for all $x \in I, k \in \mathbb{N}$, we have

$$\int_I \beta d\alpha \leq \int_I \phi_k d\alpha \leq \int_I f_k d\alpha$$

for all $k \in \mathbb{N}$. Hence

$$\int_I \beta d\alpha \leq \liminf_{k\to\infty} \int_I \phi_k d\alpha \leq \liminf_{k\to\infty} \int_I f_k d\alpha$$

Now, (ϕ_k) is an increasing sequence which converges to $\phi = \liminf f_k$ on I, and so the increasing sequence $(\int_I \phi_k d\alpha)$ is convergent, and therefore bounded. We can now apply the Monotone Convergence Theorem to conclude that $\phi = \lim \phi_k = \liminf f_k \in GRS(I, \alpha)$ and

$$\int_I \phi d\alpha = \lim_{k\to\infty} \int_I \phi_k d\alpha \in \mathbb{R}$$

Since we assumed that $\liminf \int_I f_k d\alpha < \infty$, we conclude that

$$-\infty < \int_I \liminf_{k\to\infty} f_k d\alpha \leq \liminf_{k\to\infty} \int_I f_k d\alpha < \infty$$

□

Theorem 40. *(Dominated Convergence Theorem) Let* $\alpha : I \to \mathbb{R}$ *be an increasing function. Let* (f_k) *be a sequence in* $GRS(I, \alpha)$ *with* $f(x) := \lim f_k(x)$

for all $x \in I$*. Suppose that there exist functions* $\beta, w \in GRS(I, \alpha)$ *such that* $\beta(x) \leq f_k(x) \leq w(x)$ *for all* $x \in I, k \in \mathbb{N}$*. Then* $f \in GRS(I, \alpha)$ *and*

$$\int_I f d\alpha = \lim_{k \to \infty} \int_I f_k d\alpha$$

Proof. We have $f(x) = \lim f_k(x) = \liminf f_k(x) \in \mathbb{R}$ for all $x \in I$. Since $\beta(x) \leq f_k(x) \leq w(x)$ for all $x \in I, k \in \mathbb{N}$ we have

$$\int_I \beta d\alpha \leq \int_I f_k d\alpha \leq \int_I w d\alpha \quad \forall k \in \mathbb{N}$$

So $\liminf \int_I f_k d\alpha \in \mathbb{R}$ and $\limsup \int_I f_k d\alpha \in \mathbb{R}$. Now, by Fatou's Lemma, $f \in GRS(I, \alpha)$ and $\int_I f \leq \liminf \int_I f_k d\alpha$. We now apply Fatou's Lemma to the sequence $(-f_k)$. Note that for a set A, we have $\liminf(-A) = -\limsup(A)$. So

$$-\int_I f d\alpha = \int_I (-f) d\alpha \leq \liminf_{k \to \infty} \int_I (-f_k) d\alpha = -\limsup_{k \to \infty} \int_I f_k d\alpha$$

So we conclude that

$$\limsup_{k \to \infty} \int_I f_k d\alpha \leq \int_I f d\alpha$$

Combining the consequences of Fatou's Lemma for the sequences (f_k) and $(-f_k)$, we obtain

$$\int_I f d = \lim_{k \to \infty} \int_I f_k d\alpha$$

□

Remark. Note that if at least one of β, w, f_n is absolutely integrable then the limit function f is also absolutely integrable.

Theorem 41. *(Mean Convergence Theorem) Let* $\alpha : I \to \mathbb{R}$ *be an increasing function. Let* (f_k) *be a sequence of functions with* $f_k \in GRS(I, \alpha)$ *for all* $k \in \mathbb{N}$ *and*

$$f(x) = \lim_{k \to \infty} f_k(x) \quad \forall x \in I$$

Suppose that there exist functions $\beta, w \in GRS(I, \alpha)$ such that $\beta(x) \le f_k(x) \le w(x)$ for all $x \in I$ and all $k \in \mathbb{N}$. Then the function $f - f_k$ is absolutely integrable for all $k \in \mathbb{N}$ and

$$\lim_{k\to\infty} \int_I |f - f_k| d\alpha = 0$$

Proof. Application of the Dominated Convergence Theorem shows that $f \in GRS(I, \alpha)$. Now since $f, f_k \in GRS(I, \alpha)$ we have $f - f_k \in GRS(I, \alpha)$ for all $k \in \mathbb{N}$. We take the limit as $k \to \infty$ of $\beta(x) \le f_k(x) \le w(x)$ to obtain

$$\beta(x) \le f(x) \le w(x) \quad \forall x \in I$$

Now, we also have $-w(x) \le -f_k(x) \le -\beta(x)$ and so

$$-w(x) + \beta(x) \le f(x) - f_k(x) \le w(x) - \beta(x)$$

therefore $-(w - \beta) \le f - f_k \le w - \beta$. Hence $0 \le |f - f_k| \le w - \beta$. Now, since $w - \beta$ is integrable and positive, it is absolutely integrable. By the Comparison Test, we conclude that $f - f_k$ is absolutely integrable. We now apply the Dominated Convergence Theorem to $g_k := |f - f_k|, \beta_1 = 0$ and $w_1 = w - \beta$. Note that $\lim_{k\to\infty} |f - f_k| = 0$. We conclude that

$$\lim_{k\to\infty} \int_I g_k d\alpha = \int_I \lim_{k\to\infty} g_k d\alpha = 0$$

and hence

$$\lim_{k\to\infty} \int_I |f - f_k| d\alpha = 0$$

□

Definition 24. Let $\alpha : I \to \mathbb{R}$ be an integrator function. A collection $\mathcal{F} \subseteq GRS(I, \alpha)$ is **equi-integrable** on I if for all $\varepsilon > 0$ there exists a gauge δ_ε on I such that if P is any δ_ε-fine partition of I and $f \in \mathcal{F}$ then

$$|S(f, P, \alpha) - \int_I f d\alpha| < \varepsilon$$

Theorem 42. *(Gordon's Theorem) Let $\alpha : I \to \mathbb{R}$ be an increasing function. Suppose that $(f_k) \in GRS(I, \alpha)$ and $f(x) = \lim_{k\to\infty} f_k(x)$ for all $x \in I$. Then $f \in GRS(I, \alpha)$ and*

$$\int_I f d\alpha = \lim_{k\to\infty} \int_I f_k d\alpha$$

if and only if for all $\varepsilon > 0$ there exists a gauge δ_ε on I such that if P is δ_ε-fine then there exists $K_P \in \mathbb{N}$ such that if $k \geq K_P$ then

$$|S(f_k, P, \alpha) - \int_I f_k d\alpha| < \varepsilon$$

Note that if (f_k) is equi-integrable then for all $\varepsilon > 0$ there exists a gauge δ_ε on I such that if P is δ_ε-fine, (and taking $K_P = 1$), then if $k \geq K_P$ we have $|S(f_k, P, \alpha) - \int_I f_k d\alpha| < \varepsilon$. So as a consequence of Gordon's Theorem, we have the Equi-Integrability Theorem:

Theorem 43. *If $(f_k) \in GRS(I, \alpha)$ is equi-integrable on I and $f(x) = \lim_{k\to\infty} f_k(x)$ for all $x \in I$ then $f \in GRS(I, \alpha)$ and*

$$\int_I f d\alpha = \lim_{k\to\infty} \int_I f_k d\alpha$$

Proof. (Of Gordon's Theorem)
For ($\Leftarrow$): We will first show that $(\int_I f_k d\alpha)$ is a Cauchy sequence. Fix $\varepsilon > 0$. By the hypotheses of the theorem, there exists $K_P \in \mathbb{N}$ such that if $k \geq K_P$ then $|S(f_k, P, \alpha) - \int_I f_k d\alpha| < \frac{\varepsilon}{[\alpha(b)-\alpha(a)+3]}$. Now, P contains finitely many tags $\{t_1, \ldots, t_n\}$ and we have $f(t) = \lim f_k(t)$ for all $t \in I$. So there exists an integer $K_\varepsilon \geq K_P$ such that if $h, k \geq K_\varepsilon$ then $|f_k(t_i) - f_h(t_i)| < \frac{\varepsilon}{[\alpha(b)-\alpha(a)+3]}$.

So

$$\begin{aligned}
&|S(f_k,P,\alpha)-S(f_h,P,\alpha)| \\
&= \left|\sum_{i=1}^{n} f_k(t_i)[\alpha(x_i)-\alpha(x_{i-1})] - \sum_{i=1}^{n} f_h(t_i)[\alpha(x_i)-\alpha(x_{i-1})]\right| \\
&\leq \sum_{i=1}^{n} |f_k(t_i)-f_h(t_i)|[\alpha(x_i)_\alpha(x_{i-1})] \\
&< \frac{\varepsilon}{[\alpha(b)-\alpha(a)+3]}[\alpha(b)-\alpha(a)]
\end{aligned}$$

Now let $h\to\infty$: We have

$$|S(f_k,P,\alpha)-S(f,P,\alpha)| \leq \frac{\varepsilon}{[\alpha(b)-\alpha(a)+2]}[\alpha(b)-\alpha(a)]$$

for $k\geq K_\varepsilon$. If $h,k\geq K_\varepsilon$ then we have

$$\begin{aligned}
\left|\int_I f_k d\alpha - \int_I f_h d\alpha\right| &\leq \left|\int_I f_k d\alpha - S(f_k,P,\alpha)\right| \\
&+|S(f_k,P,\alpha)-S(f_h,P,\alpha)| \\
&+\left|S(f_k,P,\alpha)-\int_I f_h d\alpha\right| \\
&< \frac{\varepsilon}{[\alpha(b)-\alpha(a)+3]} + \frac{\varepsilon}{[\alpha(b)-\alpha(a)+3]}[\alpha(b)-\alpha(a)] \\
&+ \frac{\varepsilon}{[\alpha(b)-\alpha(a)+3]} \\
&= \frac{\varepsilon}{[\alpha(b)-\alpha(a)+3]}[\alpha(b)-\alpha(a)+2]
\end{aligned}$$

Since $\varepsilon>0$ was arbitrary, we conclude that $(\int_I f_k d\alpha)$ is a Cauchy sequence, and hence is convergent to $A\in\mathbb{R}$. Now let $h\to\infty$: we have

$$\left|\int_I f_k d\alpha - A\right| \leq \frac{\varepsilon}{[\alpha(b)-\alpha(a)+3]}[\alpha(b)-\alpha(a)+2]$$

for $k\geq K_\varepsilon$.

Claim: $f\in GRS(I,\alpha)$ with integral A.

Proof of Claim: If P is a δ_ε-fine partition of I and $k \geq K_\varepsilon \geq K_P$ then

$$\begin{aligned}
|S(f,P,\alpha) - A| &\leq |S(f,P,\alpha) - S(f_k,P,\alpha)| + |S(f_k,P,\alpha) - \int_I f_k d\alpha| \\
&+ |\int_I f_k d\alpha - A| \\
&< \frac{\varepsilon}{[\alpha(b) - \alpha(a) + 3]}[\alpha(b) - \alpha(a) + 2] \\
&+ \frac{\varepsilon}{[\alpha(b) - \alpha(a) + 3]} \\
&+ \frac{\varepsilon}{[\alpha(b) - \alpha(a) + 3]}[\alpha(b) - \alpha(a) + 2] \\
&= \varepsilon
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $f \in GRS(I, \alpha)$ with integral A.

For ($\Rightarrow$): Fix $\varepsilon > 0$. Then there exists $M_\varepsilon \in \mathbb{N}$ such that if $k \geq M_\varepsilon$ then

$$\left| \int_I f d\alpha - \int_I f_k d\alpha \right| < \frac{\varepsilon}{3}$$

If $f \in GRS(I, \alpha)$ then there exists a gauge δ_ε such that if P is δ_ε-fine then

$$\left| S(f,P,\alpha) - \int_I f d\alpha \right| < \frac{\varepsilon}{3}$$

We choose $K_P \geq M_\varepsilon$ such that if $k \geq K_P$ then

$$|f_k(t_i) - f(t_i)| < \frac{\varepsilon}{3[\alpha(b) - \alpha(a)]}$$

for all $i = 1, \ldots, n$. This is possible since by $f_k(t_i) \to f(t_i)$, and since there are finitely many i, if we can find a suitable K_P for each i then an integer that will satisfy the requirements for all i can be found by taking the maximum of these values. So

$$\begin{aligned}
|S(f_k,P,\alpha)-S(f,P,\alpha)| &= \left|\sum_{i=1}^{n}[f_k(t_i)-f(t_i)][\alpha(x_i)-\alpha(x_{i-1})]\right| \\
&\leq \sum_{i=1}^{n}|f_k(t_i)-f(t_i)|[\alpha(x_i)-\alpha(x_{i-1})] \\
&< \sum_{i=1}^{n}\frac{\varepsilon}{3[\alpha(b)-\alpha(a)]}[\alpha(x_i)-\alpha(x_{i-1})] \\
&= \frac{\varepsilon}{3[\alpha(b)-\alpha(a)]}[\alpha(b)-\alpha(a)] \\
&= \frac{\varepsilon}{3}
\end{aligned}$$

Therefore if $k \geq K_P \geq M_\varepsilon$ then we have

$$\begin{aligned}
|S(f_k,P,\alpha)-\int_I f_k d\alpha| &\leq |S(f_k,P,\alpha)-S(f,P,\alpha)| + |S(f,P,\alpha)-\int_I f d\alpha| \\
&+ \left|\int_I f d\alpha - \int_I f_k d\alpha\right| \\
&< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} \\
&= \varepsilon
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we conclude that for $k \geq K_P$ we have

$$\left|S(f_k,P,\alpha)-\int_I f_k d\alpha\right| < \varepsilon$$

□

Chapter 7

The Simplification Theorem

The Simplification Theorem is an original result which converts between Generalised Riemann-Stieltjes integrals and Riemann or Generalised Riemann integrals. Not only does this provide an easier way of calculating Generalised Riemann-Stieltjes integrals, it also gives a simple geometric interpretation as the area under a suitable function. This theorem is based on similar work in [2], [3] and [4].

Theorem 44. *(Simplification Theorem) Let $\alpha : [a, b] \to \mathbb{R}$ be an increasing function with $\alpha \in C^1([a, b])$. Let $f : [a, b] \to \mathbb{R}$ be a continuous function with $f \in GRS([a, b], \alpha)$. Then*

$$\int_a^b f(x)d\alpha(x) = \int_a^b f(x)\alpha'(x)dx$$

where the integral on the left is a Generalised Riemann-Stieltjes integral, and the integral on the right is a Generalised Riemann integral.

Remark. Note that the integral on the right exists, since all continuous functions are Generalised Riemann integrable.

Proof. We define $g(x) := f(x)\alpha'(x)$. Suppose that $P = \{(I_i, t_i)\}$ is a partition of I, and consider the Riemann sum

$$S(g,P) = \sum_{k=1}^{n} g(t_k)(x_k - x_{k-1})$$
$$= \sum_{k=1}^{n} f(t_k)\alpha'(t_k)(x_k - x_{k-1})$$

Now we consider the following Riemann-Stieltjes sum

$$S(f,P,\alpha) = \sum_{k=1}^{n} f(t_k)[\alpha(x_k) - \alpha(x_{k-1})]$$

We apply the Mean Value Theorem to the function α to show that

$$\alpha(x_k) - \alpha(x_{k-1}) = \alpha'(v_k)(x_k - x_{k-1})$$

for some $v_k \in (x_{k-1}, x_k)$. Substituting into the Riemann-Stieltjes sum, we obtain

$$S(f,P,\alpha) = \sum_{k=1}^{n} f(t_k)\alpha'(v_k)(x_k - x_{k-1})$$

So we have

$$S(f,P,\alpha) - S(g,P) = \sum_{k=1}^{n} f(t_k)[\alpha'(v_k) - \alpha'(v_k)](x_k - x_{k-1})$$

Since f is a continuous function on a closed interval, it is bounded, and so there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in I$. Since α' is continuous on $[a,b]$, it is also uniformly continuous on $[a,b]$. So if we have fixed $\varepsilon > 0$ then there exists $\delta_\varepsilon^c > 0$ such that

$$0 \leq |x - y| \leq 2\delta_\varepsilon^c \Rightarrow |\alpha'(x) - \alpha'(y)| < \frac{\varepsilon}{2M(b-a)}$$

Now, since $f \in GRS([a,b],\alpha)$ there exists a gauge γ_ε on I such that if P is γ_ε-fine then $|S(f,P,\alpha) - \int_a^b f d\alpha| < \frac{\varepsilon}{2}$. We define the function $\delta_\varepsilon(t) := \min\{\delta_\varepsilon^c, \gamma_\varepsilon(t)\}$ for $t \in I$. Now δ_ε is a gauge, and if a partition P is δ_ε fine it is also δ_ε^c-fine and γ_ε-fine. So if P is δ_ε-fine then

$$|S(f,P,\alpha) - \int_a^b f d\alpha| < \frac{\varepsilon}{2}$$

If P is δ_ε-fine, then for each interval $[x_{k-1}, x_k]$ in P we have

$$\begin{aligned}[x_{k-1}, x_k] &\subseteq [t_i - \delta_\varepsilon(t_i), t_i + \delta_\varepsilon(t_i)] \\ &\subseteq [t_i - \delta_\varepsilon^c, t_i + \delta_\varepsilon^c]\end{aligned}$$

and therefore $x_k - x_{k-1} = |x_k - x_{k-1}| \leq 2\delta_\varepsilon^c$. Now if $v_k \in (x_{k-1}, x_k)$, we have $|v_k - t_k| < |x_k - x_{k-1}| \leq 2\delta_\varepsilon^c$. Then by uniform continuity of α', we have

$$|\alpha'(v_k) - \alpha'(t_k)| < \frac{\varepsilon}{2M(b-a)}$$

Now we have

$$\begin{aligned}|S(f, P, \alpha) - S(g, P)| &= \sum_{k=1}^{n} f(t_k)[\alpha'(v_k) - \alpha'(t_k)](x_k - x_{k-1}) \\ &\leq \sum_{k=1}^{n} M \frac{\varepsilon}{2M(b-a)}(x_k - x_{k-1}) \\ &= \frac{\varepsilon}{2(b-a)} \sum_{k=1}^{n}(x_k - x_{k-1}) \\ &= \frac{\varepsilon}{2(b-a)}(b-a) \\ &= \frac{\varepsilon}{2}\end{aligned}$$

and therefore, using the Triangle Inequality,

$$\begin{aligned}|S(P, g) - \int_a^b f d\alpha| &\leq |S(P, g) - S(f, P, \alpha)| + |S(f, P, \alpha) - \int_a^b f d\alpha| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $g = f\alpha'$ is Generalised Riemann integrable with

$$\int_a^b f(x) d\alpha(x) = \int_a^b f(x)\alpha'(x) dx$$

□

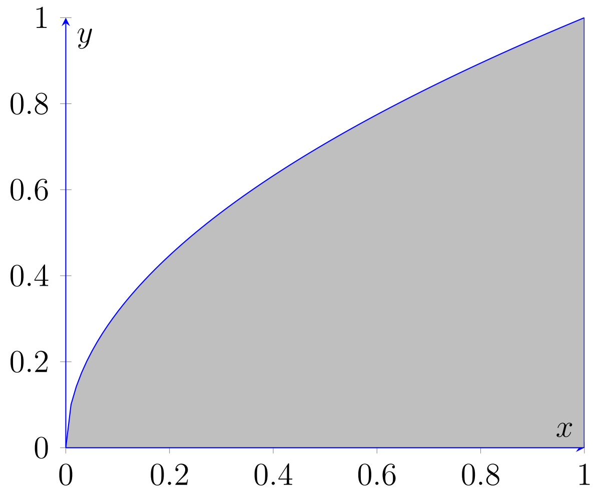
Remark. How should the Riemann-Stieltjes integral be visualised? The geometric interpretation of the Riemann integral of the function $f(x)$ from $x = a$ to $x = b$ is the area under the curve $y = f(x)$ between those endpoints. The Riemann-Stieltjes integral in Theorem 44 can be interpreted geometrically as the area below the curve $(x, y) = (\alpha(t), f(t))$ for $t \in [a, b]$. This allows us to interpret the integral as the familiar sum of rectangles.

Let us see some examples of this process.

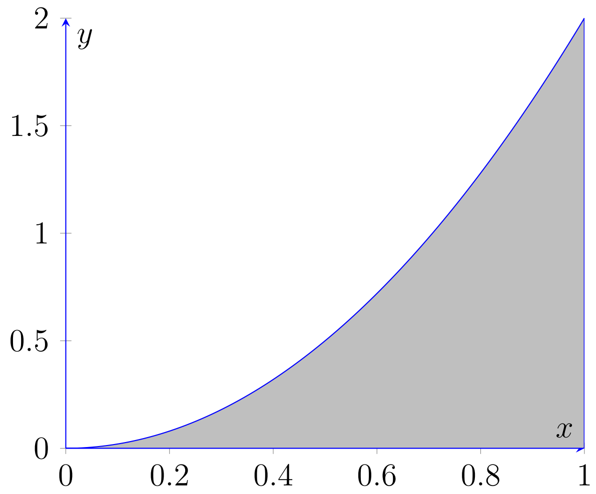
Example 16. Suppose $[a, b] = [0, 1]$, $\alpha(x) = x^2$ and $f(x) = x$. We calculate $\int_a^b f(x)d\alpha(x)$ using the Simplification Theorem. We have

$$\begin{aligned}\int_0^1 x dx^2 &= \int_0^1 x\alpha'(x)dx \\ &= \int_0^1 2x^2 dx \\ &= \frac{2}{3}\end{aligned}$$

To geometrically interpret the integral on the left hand side, we look at the area under the graph $(x, y) = (t^2, t)$ for $0 \leq t \leq 1$:



To geometrically interpret the integral on the right hand side, we look at the area under the graph $y = 2x^2$ for $0 \leq x \leq 1$:

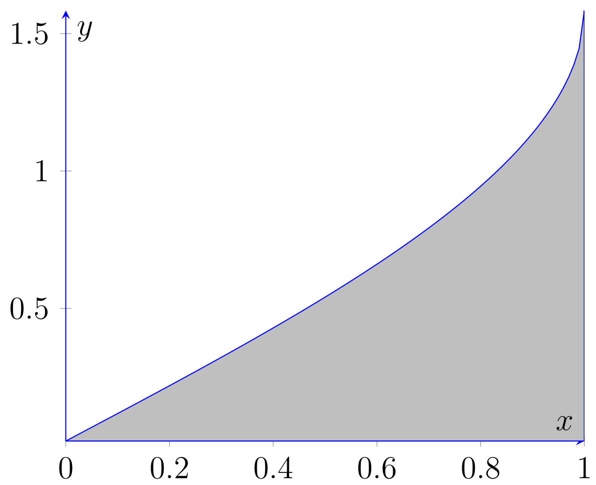


Both of the shaded regions have an area of $\frac{2}{3}$.

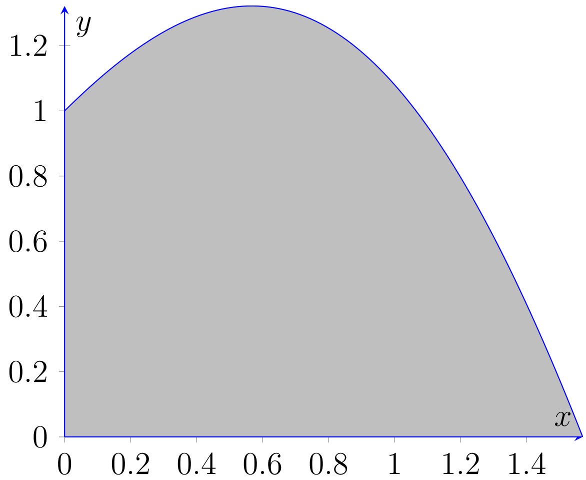
Example 17. Suppose $[a,b]=[0,\frac{\pi}{2}]$, $f(x)=x+1$, and $\alpha(x)=\sin(x)$. We calculate $\int_a^b f(x)d\alpha(x)$ using the Simplification Theorem. We have

$$\begin{aligned}\int_0^{\frac{\pi}{2}}(x+1)d\sin(x) &= \int_0^{\frac{\pi}{2}} x+1\alpha'(x)dx\\ &= \int_0^{\frac{\pi}{2}}(x+1)\cos(x)dx\\ &= \frac{\pi}{2}\end{aligned}$$

To geometrically interpret the integral on the left hand side, we look at the area under the graph $(x,y)=(\sin(t),t+1)$ for $0\le t\le\frac{\pi}{2}$:



To geometrically interpret the integral on the right hand side, we look at the area under the graph $y = (x+1)\cos(x)$ for $0 \leq t \leq \frac{\pi}{2}$:



Chapter 8

Existence Results

In this chapter, we prove a variety of results of an important type - existence theorems. As with other types of integration such as Generalised Riemann and Lebesgue, we start by examining the integrability of step functions and then build on that to prove integrability of more complicated functions.

Our first theorem proves the integrability of the simplest possible form of step function.

Definition 25. A function $s : I \to \mathbb{R}$ is a step function if there exist $a_1, \ldots, a_n \in \mathbb{R}$ and a partition $\{[c_{i-1}, c_i]\}$ of I such that

$$s(x) = a_i \quad \text{for} \quad x \in (c_{i-1}, c_i)$$

and $s(x) = 1$ for $x = c_i$, where $i = 1, \ldots, n$.

Remark. The value of s at the partition points does not matter for integration; it could be set to any number.

Theorem 45. *Suppose that $\alpha : I \to \mathbb{R}$ is Lipschitz continuous with constant*

K. Let the function $s_j : I \to \mathbb{R}$ defined by

$$s_j(x) = \begin{cases} a_j & \textit{if} \quad x \in (c_{j-1}, c_j) \\ 1 & \textit{otherwise} \end{cases}$$

where $a_j \neq 0 \in \mathbb{R}$ and $(c_{j-1}, c_j) \subseteq (a, b)$. Then $s_j \in GRS(I, \alpha)$ and

$$\int_a^b s_j d\alpha = a_j[\alpha(c_j) - \alpha(c_{j-1})]$$

Proof. Fix $\varepsilon > 0$. We define a gauge $\delta_\varepsilon : I \to (0, \infty)$ by

$$\delta_\varepsilon(t) = \begin{cases} \frac{1}{2}\text{dist}(t, \{c_{j-1}, c_j\}) & \text{if} \quad t \notin \{c_{j-1}, c_j\} \\ \delta & \text{if} \quad t \in \{c_{j-1}, c_j\} \end{cases}$$

where we can choose the value of $\delta \in (0, \infty)$ later. So c_{j-1} and c_j are tags of any subintervals containing those points. We split such intervals into two subintervals with partition point (and tag) given by c_{j-1} or c_j. This procedure does not change the value of the Riemann-Stieltjes sum. Now c_{j-1} is the tag for subintervals $[x_{r-1}, x_r]$ and $[x_r, x_{r+1}]$ where $c_{j-1} = x_r = t_r = t_{r+1}$, and c_j is the tag for subintervals $[x_{s-1}, x_s]$ and $[x_s, x_{s+1}]$ where $c_j = x_s = t_s = t_{s+1}$. The non-zero terms in the Riemann-Stieltjes sum are those which correspond to tags $t_{r+2}, \dots, t_{s-1}$. So

$$S(s_j, P, \alpha) = \sum_{i=r+2}^{i=s-1} a_j[\alpha(x_i) - \alpha(x_{i-1})] = a_j[\alpha(x_{s-1}) - \alpha(x_{r+1})]$$

We have

$$\alpha(x_{s-1}) = \alpha(c_j) - [\alpha(x_{s-1}) - \alpha(c_j)]$$

and

$$\alpha(x_{r+1}) = \alpha(c_{j-1}) - [\alpha(x_{r+1}) - \alpha(c_{j-1})]$$

So we can rewrite the Riemann-Stieltjes sum as

$$S(s_j, P, \alpha) = a_j[\alpha(c_j) - \alpha(c_{j-1})] + a_j[\alpha(x_{s-1}) - \alpha(c_j)] - a_j[\alpha(x_{r+1}) - \alpha(c_{j-1})]$$

Now since α is Lipschitz, for $y_1, y_2 \in I$ we have $|\alpha(y_1) - \alpha(y_2)| \leq K|y_1 - y_2|$and therefore $|\alpha(y_1) - \alpha(y_2)| < (K+1)|y_1 - y_2|$. If P is δ_ε-fine then $|c_j - x_{s-1}| < 2\delta$ and $|x_{r+1} - c_{j-1}| < 2\delta$. Hence $|\alpha(c_j) - \alpha(x_{s-1})| < 2(K+1)\delta$ and $|\alpha(x_{r+1}) - \alpha(c_{j-1})| < 2(K+1)\delta$. Now we have

$$\begin{aligned}
&|S(s_j, P, \alpha) - a_j[\alpha(c_j) - \alpha(c_{j-1})]| \\
=&|a_j[\alpha(x_{s-1}) - \alpha(c_j)] - a_j[\alpha(x_{r+1}) - \alpha(c_{j-1})]| \\
\leq&|a_j[\alpha(x_{s-1}) - \alpha(c_j)]| + |a_j[\alpha(x_{r+1}) - \alpha(c_{j-1})]| \\
<&2|a_j|(K+1)\delta + 2|a_j|(K+1)\delta \\
=&4|a_j|(K+1)\delta
\end{aligned}$$

Now $a_j \neq 0$ so we can choose any value of δ such that

$$\delta \leq \frac{\varepsilon}{4|a_j|(K+1)}$$

So for such δ we have $|S(s_j, P, \alpha) - a_j[\alpha(c_j) - \alpha(c_{j-1})]| < \varepsilon$, and since $\varepsilon > 0$ was arbitrary we conclude that $s_j \in GRS(I, \alpha)$ and

$$\int_a^b s_j d\alpha = a_j[\alpha(c_j) - \alpha(c_{j-1})]$$

□

Corollary 8. *Suppose that $\alpha : I \to \mathbb{R}$ is Lipschitz continuous with constant K. Then all step functions $s : I \to \mathbb{R}$ are integrable with respect to α. Furthermore, if s is defined as in Definition 25, we have*

$$\int_a^b s d\alpha = \sum_{i=1}^{n} a_i[\alpha(c_i) - \alpha(c_{i-1})]$$

Proof. We have, for s_i as in the previous theorem,

$$s = s_1 + \cdots + s_n$$

Since s_i is integrable for each $i \in \{1, \ldots, n\}$, by linearity we conclude that s is integrable and that

$$\int_a^b s d\alpha = \sum_{i=1}^n \int_a^b s_i d\alpha = \sum_{i=1}^n a_i[\alpha(c_i) - \alpha(c_{i-1})]$$

□

Definition 26. A function $f : I \to \mathbb{R}$ is regulated if for all $\varepsilon > 0$ there exists a step function $s_\varepsilon : I \to \mathbb{R}$ such that

$$|f(x) - s_\varepsilon(x)| < \varepsilon \quad \text{for all} \quad x \in I$$

Remark. By setting $\varepsilon = \frac{1}{n}$, we see that a function f is regulated if and only if there exists a sequence (s_n) of step functions which converges uniformly to f on I.

Theorem 46. *Suppose $f : I \to \mathbb{R}$ is regulated, and $\alpha : I \to \mathbb{R}$ is Lipschitz continuous. Then $f \in GRS(I, \alpha)$.*

Proof. Fix $\varepsilon > 0$. Let s_ε be a step function such that

$$|f(x) - s_\varepsilon(x)| < \varepsilon \quad \text{for all} \quad x \in I$$

We define functions $\phi_\varepsilon, \psi_\varepsilon : I \to \mathbb{R}$ by

$$\phi_\varepsilon(x) := s_\varepsilon(x) - \varepsilon, \psi_\varepsilon(x) := s_\varepsilon(x) + \varepsilon$$

for all $x \in I$. So the functions $\phi_\varepsilon, \psi_\varepsilon$ are step functions and are therefore integrable on I. Also, note that $\phi_\varepsilon(x) \le f(x) \le \psi_\varepsilon(x)$ for all $x \in I$, and

$$\int_a^b (\psi_\varepsilon - \phi_\varepsilon) d\alpha = \int_a^b 2\varepsilon d\alpha = 2\varepsilon[\alpha(b) - \alpha(a)]$$

So, applying the Squeeze Theorem, we conclude that f is integrable on I with respect to α. □

The following theorem allows us an easier way to prove that a function is regulated and thus integrable.

Theorem 47. *(Characterisation of Regulated Functions) A function $f : I \to \mathbb{R}$ is regulated if and only if it has all of its one-sided limits at every point in I.*

Proof. For ($\Rightarrow$): Firstly, every step function has one-sided limits at every point in I. Now, fix $c \in [a, b)$ and $\varepsilon > 0$. Let $s_\varepsilon : I \to \mathbb{R}$ be a step function with the property that

$$|f(x) - s_\varepsilon(x)| < \frac{\varepsilon}{2} \quad \forall x \in I$$

Since $\lim_{x\to c^+} s_\varepsilon(x)$ exists, and s_ε is a step function, there exists $\delta_\varepsilon(c) > 0$ such that

$$x, y \in (c, c + \delta_\varepsilon(c)) \Rightarrow s_\varepsilon(x) = s_\varepsilon(y)$$

So if $x, y \in (c, c + \delta_\varepsilon(c))$ then

$$\begin{aligned} |f(x) - f(y)| &\le |f(x) - s_\varepsilon(x)| + |s_\varepsilon(x) - s_\varepsilon(y)| + |s_\varepsilon(y) - f(y)| \\ &< \frac{\varepsilon}{2} + 0 + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

Now $\varepsilon > 0$ was arbitrary, so by the Cauchy Criterion, the limit

$$\lim_{x\to c^+} f(x)$$

exists. We can use a similar argument to show that left-hand limits exist. For ($\Leftarrow$): Suppose that all one-sided limits of f exist at every point of I. Using the Cauchy criterion, we know that for all $\varepsilon > 0$ there is a gauge δ_ε on I such that if $t \in I$ and y_1, y_2 are both in either $[t - \delta_\varepsilon(t), t)$ or $(t, t + \delta_\varepsilon(t)]$ then

$$|f(y_1) - f(y_2)| < \varepsilon$$

Let $P = \{([x_{i-1}, x_i], t_i)\}$ be a δ_ε-fine partition of I. We define a function $s_\varepsilon : I \to \mathbb{R}$ as follows. If z is one of the partition points or tags then we set $s_\varepsilon(z) := f(z)$. If $x \in (x_{i-1}, t_i) \subseteq [t_i - \delta_\varepsilon(t_i), t_i)$ then we define

$$s_\varepsilon(x) := f\left(\frac{1}{2}(x_{i-1} + t_i)\right)$$

so that we have

$$|f(x) - s_\varepsilon(x)| = |f(x) - f\left(\frac{1}{2}(x_{i-1} + t_i)\right)| < \varepsilon$$

Similarly, if $x \in (t_i, x_i) \subseteq (t_i, t_i + \delta_\varepsilon(t_i))$ then we define

$$s_\varepsilon(x) := f\left(\frac{1}{2}(t_i + x_i)\right)$$

so that we have

$$|f(x) - s_\varepsilon(x)| = |f(x) - f\left(\frac{1}{2}(t_i + x_i)\right)| < \varepsilon$$

Hence s_ε is a step function satisfying

$$|f(x) - s_\varepsilon(x)| < \varepsilon \quad \forall x \in I$$

Since $\varepsilon > 0$ is arbitrary, we conclude that f is a regulated function. □

The next results are consequences of the above characterisation.

Theorem 48. *If $f : I \to \mathbb{R}$ is continuous and $\alpha : I \to \mathbb{R}$ is Lipschitz continuous, then $f \in GRS(I, \alpha)$.*

Proof. Since f is continuous, it has a limit at every point of I, and so using the Characterisation Theorem, we conclude that f is regulated. Hence it is integrable. □

Theorem 49. *If $f : I \to \mathbb{R}$ is monotone and $\alpha : I \to \mathbb{R}$ is Lipschitz continuous, then $f \in GRS(I, \alpha)$.*

Proof. Since f is monotone, it has one-sided limits at each point in I, so using the Characterisation Theorem, it is regulated. Hence f is integrable. $\square$

The following result examines the integrability of products of functions. So far we have only considered the product of a function with a constant (as part of the proof that the collection of integrable functions is a vector space), but with the existence results in this section we can now consider products of functions. A similar approach is taken concerning the Generalised Riemann integrability of products.

Theorem 50. *Suppose that $f : I \to \mathbb{R}$ is bounded below, $g : I \to \mathbb{R}$ is regulated and $\alpha : I \to \mathbb{R}$ is Lipschitz continuous and increasing. Then $fg \in GRS(I, \alpha)$.*

Proof. It is sufficient to consider the case that $f(x) \geq 0$ for all $x \in I$. If $s : I \to \mathbb{R}$ is a step function, then $fs \in GRS(I, \alpha)$. Now, let $A \in \mathbb{R}$ be such that

$$A > \int_a^b f d\alpha > 0$$

and fix $\varepsilon > 0$. Since g is regulated, there exists a step function $s_\varepsilon : I \to \mathbb{R}$ such that

$$|g(x) - s_\varepsilon(x)| < \frac{\varepsilon}{2A} \quad \forall x \in I$$

We define functions $\phi_\varepsilon, \psi_\varepsilon : I \to \mathbb{R}$ by

$$\phi_\varepsilon(x) := f(x)[s_\varepsilon(x) - \frac{\varepsilon}{2A}], \psi_\varepsilon(x) := f(x)[s_\varepsilon(x) + \frac{\varepsilon}{2A}] \quad \forall x \in I$$

So both ϕ_ε and ψ_ε are integrable and

$$\phi_\varepsilon(x) \leq f(x)g(x) \leq \psi_\varepsilon(x) \quad \forall x \in I$$

Furthermore, we have

$$\int_a^b (\psi_\varepsilon - \phi_\varepsilon) d\alpha = \frac{\varepsilon}{A} \int_a^b f d\alpha < \varepsilon$$

Hence, by the Squeeze Theorem, we conclude that $fg \in GRS(I, \alpha)$. $\square$

Chapter 9

Hake's Theorem and Infinite Intervals

In this chapter we see an analogue of Hake's Theorem, a result which concerns improper integrability. As in the context of the Generalised Riemann integral, it can be shown that integrability and improper integrability are equivalent, or, in other words, the integral cannot be extended by taking limits. This result is important both in calculation and in working with infinite intervals. To start, we need to review some results from topology.

Definition 27. Suppose $E \subseteq [a, b]$. A collection $\mathcal{F}$ of non-degenerate closed subintervals of $[a-1, b+1]$ is a **Vitali covering** of E if for all $x \in E$ and all $s > 0$ there exists an interval $J \in \mathcal{F}$ such that $x \in J$ and $0 < l(J) < s$.

Remark. If $\mathcal{F}$ is a Vitali covering for E then every $x \in E$ is contained in infinitely many intervals in $\mathcal{F}$.

Example 18. Let $E = [a, b] = [0, 1]$. The collection of closed balls $B(\frac{1}{n}, r)$ where $n \in \mathbb{N}$ and $r \in \mathbb{Q} \cap [0, 1]$ is a Vitali covering of E.

Theorem 51. *(Vitali Covering Theorem) Let $E \subseteq [a,b]$ and let $\mathcal{F}$ be a Vitali covering of E. Given $\varepsilon > 0$, there exist disjoint intervals $I_1, \ldots, I_p \in \mathcal{F}$ and $\{J_i : i \geq p+1\}$, a countable collection of closed intervals in $\mathbb{R}$ such that*

$$E - \bigcup_{i=1}^{p} I_i \subseteq \bigcup_{i=p+1}^{\infty} J_i \quad \text{and} \quad \sum_{i=p+1}^{\infty} l(J_i) < \varepsilon$$

and hence

$$E \subseteq \bigcup_{i=1}^{p} I_i \bigcup \bigcup_{i=p+1}^{\infty} J_i$$

Proof. Fix $I_1, \ldots, I_r \in \mathcal{F}$. If

$$E \subseteq \bigcup_{i=1}^{r} I_i$$

then we take $J_i = \emptyset$ for all $i \geq r+1$. Otherwise, we define

$$\mathcal{F}_r = \{I \in \mathcal{F} : \quad \text{I contains points of E and I is disjoint from} \quad I_1, \ldots, I_r\}$$

Let λ_r be the supremum of the lengths of intervals in $\mathcal{F}_r$. So we must have $\lambda_r \leq b - a + 2$ from the definition of a Vitali covering. We choose $I_{r+1} \in \mathcal{F}_r$ such that $l(I_{r+1}) \leq \frac{1}{2}\lambda_r$. By performing this procedure iteratively, we obtain a sequence of intervals (I_i) which is infinite unless E is contained in the union of some finite number, say k, in which case we set $J_{r+k} = \cdots = \emptyset$. So we suppose that we obtain an infinite sequence (I_i). The I_i are pairwise disjoint and contained in $[a-1, b+1]$ so

$$\sum_{i=1}^{\infty} l(I_i) \leq b - a + 2$$

Fix $\varepsilon > 0$. There exists $p \in \mathbb{N}$ such that

$$\sum_{i=p+1}^{\infty} l(I_i) < \frac{\varepsilon}{5}$$

We define

$$D_p := E - \bigcup_{i=1}^{p} I_i$$

Fix $x \in D_p$. Then since $\mathcal{F}$ is a Vitali covering, there exists an interval $I_x \in \mathcal{F}$ such that $x \in I_x$ and $I_x \cap I_i = \emptyset$ for all $i = 1, \ldots, p$. So $I_x \in \mathcal{F}_p$.

Claim: The interval I_x must intersect at least one I_n for $n > p$.

Proof of Claim: If $I_x \cap I_i = \emptyset$ for $i = 1, \ldots, n$ then $I_x \in \mathcal{F}_n$ and $0 < l(I_x) \leq \lambda_n$. But we have $0 \leq \lambda_n < 2l(I_{n+1})$ and so

$$\lim_{n \to \infty} = 0$$

But then it is not possible that $0 < l(I_x) \leq \lambda_n$ for all $n \in \mathbb{N}$. Let $n(x)$ be the smallest $n \in \mathbb{N}$ such that $I_x \cap I_n \neq \emptyset$. We have $n(x) > p$ by definition. Also, since $I_x \in \mathcal{F}_{n(x)-1}$, we have $l(I_x) \leq \lambda_{n(x)-1}$. Now I_x contains x and also contains a point in the interval $I_{n(x)}$. The distance from x to $x_{n(x)}$, the midpoint of $I_{n(x)}$, is given by

$$\begin{aligned} |x - x_{n(x)}| &\leq l(I_x) + \frac{1}{2} l[I_{n(x)}] \\ &< 2l[I_{n(x)}] + \frac{1}{2} l[I_{n(x)}] \\ &= \frac{5}{2} l[I_{n(x)}] \end{aligned}$$

Let $J_{n(x)}$ be the interval with the same midpoint of $I_{n(x)}$ but with $l(J_{n(x)}) = 5l[I_{n(x)}]$. We have $x \in J_{n(x)}$. For $i \geq p+1$, let the interval J_i be constructed from I_i in this way. Since $x \in D_p$ was arbitrary, we conclude that

$$E - \bigcup_{i=1}^{p} I_i = D_p \subseteq \bigcup_{i=p+1}^{\infty} J_i$$

We also have

$$\sum_{i=p+1}^{\infty} l(J_i) = 5 \sum_{i=p+1}^{\infty} l(I_i) \leq 5 \frac{\varepsilon}{5} = \varepsilon$$

□

Definition 28. If $f \in GRS(I, \alpha)$ and $u \in I$ then the function $F_u : I \to \mathbb{R}$ given by

$$F_u(x) = \int_u^x f d\alpha$$

is the **indefinite integral** of f with basepoint u. Any function that differs from F_a by a constant is an indefinite integral of f.

Definition 29. The subset $Z \subseteq I$ is a **null set** if for all $\varepsilon > 0$ there exists a countable collection of intervals $\{U_n\}$ such that

$$Z \subseteq \bigcup_{i=1}^{\infty} U_n \quad \text{and} \quad \sum_{i=1}^{\infty} l(U_n) < \varepsilon$$

The next result resembles the Fundamental Theorem, but instead concerns indefinite integrals. It has an important role in the proof of Hake's Theorem.

Theorem 52. *(Differentiation Theorem) Suppose that $f \in GRS(I, \alpha)$. Let F be an indefinite integral of f. Then there exists a null set $Z \subseteq I$ such that if $x \in I - Z$ then $F'(x)$ exists and $F'(x) = f(x)$. Hence F is also continuous on I.*

Proof. We define

$$E := \{x \in [a, b) : F'_+(x) \quad \text{either does not exist at x or is not equal to f(x)}\}$$

where $F'_+(x)$ is the right hand derivative of F at x.
Claim: E is a null set.
Proof of claim: If F has a right hand derivative at $x \in I$ then for all $\beta > 0$ there exists $s > 0$ such that if $v \in I$ is such that $x < v < x + s$ then

$$\left| \frac{F(v) - F(x)}{v - x} - f(x) \right| \leq \beta$$

If we negate this statement, we obtain: if $x \in E$ then there exists $\beta(x) > 0$ such that for all $s > 0$ there exists $v_{x,s} \in I$ with $x < v_{x,s} < x + s$ and

$$\left| \frac{F(v_{x,s}) - F(x)}{v_{x,s} - x} - f(x) \right| > \beta(x)$$

Multiplying by $|v_{x,s} - x|$, we obtain

$$|F(v_{x,s}) - F(x) - f(x)(v_{x,s} - x)| > \beta(x)|v_{x,s} - x|$$

For $n \in \mathbb{N}$, we define

$$E_n := \{x \in E : \beta(x) \geq \frac{1}{n}\}$$

Fix $\varepsilon > 0$. Since f is Generalised Riemann integrable, there exists a gauge δ_ε on I such that if a partition P is δ_ε-fine then

$$\left| S(f, P, \alpha) - \int_I f d\alpha \right| < \frac{\varepsilon}{n}$$

We define

$$\mathcal{F}_n := \{[x, v_{x,s}] : x \in E_n, 0 < s \leq \delta_\varepsilon(x)\}$$

Now $\mathcal{F}$ is a Vitali covering for E_n since

$$0 < v_{x,s} - x < s \Leftrightarrow x < v_{x,s} < x + s$$

By the Vitali Covering Theorem, there exist intervals

$$I_1 = [x_1, v_1], \dots, I_p = [x_p, v_p] \in \mathcal{F}_n$$

and a sequence $(J_i)_{p+1}^{\infty}$ of closed intervals such that

$$E_n \subseteq \bigcup_{i=1}^{p} I_i \bigcup \bigcup_{i=p+1}^{\infty} J_i \quad \text{and} \quad \sum_{i=p+1}^{\infty} l(J_i) < \varepsilon$$

We have

$$\begin{aligned}\sum_{i=1}^{p} |f(x_i)(v_i - x_i) - \int_{x_i}^{v_i} f d\alpha| &= \sum_{i=1}^{p} |f(x_i)(v_i - x_i) - [F(v_i) - F(x_i)]| \\ &> \frac{1}{n} \sum_{i=1}^{p} (v_i - x_i)\end{aligned}$$

when $\beta(x_i) \geq \frac{1}{n}$. But we have $s \leq \delta_\varepsilon(x_i)$ and so $x + s \leq x + \delta_\varepsilon(s)$, and $x_i \leq v_i \leq x_i + \delta_\varepsilon(x_i)$. So $\{(I_i, x_i)\}_{i=1}^p$ forms a subpartition of a δ_ε-fine partition of I for white

$$\left| S(f, P, \alpha) - \int_I f d\alpha \right| < \frac{\varepsilon}{n}$$

By Corollary 5 of the Saks-Henstock Lemma, we have

$$\sum_{i=1}^{p} |f(x_i)(v_i - x_i) - [F(v_i) - F(x_i)]| \leq \frac{2\varepsilon}{n}$$

Combining inequalities gives

$$\frac{1}{n}\sum_{i=1}^{p}(v_i - x_i) \leq \sum_{i=1}^{p} |f(x_i)(v_i - x_i) - [F(v_i) - F(x_i)]| \leq \frac{2\varepsilon}{n}$$

Now we multiply by n to obtain

$$\sum_{i=1}^{p}(v_i - x_i) \leq n \sum_{i=1}^{p} |f(x_i)(v_i - x_i) - \int_{x_i}^{v_i} f d\alpha| \leq 2\varepsilon$$

But E_n is contained in a countable union of intervals. By the above inequality, the total lengths of the I_i is bounded by 2ε. As part of the Vitali Covering Theorem we obtained $\sum_{i=p+1}^{\infty} l(J_i) < \varepsilon$. So E_n is contained in a countable union of intervals with total length bounded by 3ε. Since $\varepsilon > 0$ was arbitrary, we conclude that E_n is a null set. We have

$$E = \bigcup_{n=1}^{\infty} E_n$$

so E is also a null set. Similarly, the set of points in $(a, b]$ where F does not have a left hand derivative is also a null set. The set Z is the union of these two null sets, and hence is also a null set. □

Theorem 53. *(Hake's Theorem) A function $f : I \to \mathbb{R}$ is integrable if and only if there exists $A \in \mathbb{R}$ such that for all $c \in (a, b)$ the restriction of f to $[a, c]$ is integrable and*

$$\lim_{c \to b^-} \int_a^c f d\alpha = A$$

We then have

$$A = \int_a^b f d\alpha$$

Proof. For ($\Rightarrow$): If $c \in (a, b)$ then $f|_{[a,b]}$ is integrable. The indefinite integral of f with basepoint a is continuous at b (by the Differentiation Theorem), and so

$$\int_a^b f d\alpha = \lim_{c \to b^-} \int_a^c f d\alpha$$

For ($\Leftarrow$): Suppose that there exists $A \in \mathbb{R}$ such that for all $c \in (a, b)$, the restriction $f_{[a,c]}$ is integrable on $[a, c]$ and

$$\lim_{c \to b^-} \int_a^c f d\alpha = A$$

Let $(c_k)_{k=0}^{\infty}$ be a strictly increasing sequence of points with $a = c_0$ and $b = \lim_{k\to\infty} c_k$. Fix $\varepsilon > 0$. We choose $r \in \mathbb{N}$ such that

$$b - c_r \leq \frac{1}{3} \frac{\varepsilon}{|f(b)| + 1}$$

and

$$t \in [c_r, b) \Rightarrow \left| \int_a^t f d\alpha - A \right| < \frac{1}{3}\varepsilon$$

For $k \in \mathbb{N}$, let δ_k be a gauge on $I_k = [c_{k-1}, c_k]$ such that if P_k is a δ_k-fine partition of I_k then

$$\left| S(f, P_k, \alpha) - \int_{I_k} f d\alpha \right| < \frac{1}{3} \frac{\varepsilon}{2^k}$$

Without loss of generality we may assume that, by redefining the δ_k, if necessary, we have

$$\delta_1(c_0) \leq \frac{1}{2}(c_1 - c_0)$$

$$\text{For} \quad k \geq 1 : \delta_{k+1}(c_k) \leq \min\{\delta_k(c_k), \frac{1}{2}(c_k - c_{k-1}), \frac{1}{2}(c_{k+1} - c_k)\}$$

$$\text{For} \quad k \geq 1 : \delta_k(t) \leq \min\{\frac{1}{2}(t - c_k), \frac{1}{2}(c_k - t)\} \quad \text{for} \quad t \in (c_{k-1}, c_k)$$

We define a gauge on I by

$$\delta(t) = \begin{cases} \delta_k(t) & \text{if} \quad t \in [c_{k-1}, c_k) \\ b - c_r & \text{if} \quad t = b \end{cases}$$

If a partition P is δ-fine then since $b \notin I_k$ for any k, the last subinterval $[x_{n-1}, b]$ must have tag $t_n = b$. If P is δ-fine then $c_r = b - \delta(b) \leq x_{n-1}$. Let $s \in \mathbb{N}$ be the smallest positive integer such that $x_{n-1} \leq c_s$, so that we have $r \leq s$. If $k \in \{1, \ldots, s-1\}$ then c_k must be the tag for any subinterval which contains it. By splitting such intervals in two with partition point c_k, we may assume that $c_0, \ldots, c_{s-1}$ are the partition points. Let

$$Q_1 := P \cap [c_0, c_1], \ldots, Q_{s-1} := P \cap [c_{s-2}, c_{s-1}], Q_s := P \cap [c_{s-1}, x_{n-1}]$$

Now each Q_k is a δ_k-fine partition of I_k, and so we have

$$\left| S(f, Q_k, \alpha) - \int_{I_k} f d\alpha \right| < \frac{1}{3} \frac{\varepsilon}{2^k}$$

Since Q_S is δ_s-fine subpartition of I_s, we have

$$\left| S(f, Q_s, \alpha) - \int_{c_{s-1}}^{x_{n-1}} f d\alpha < \frac{1}{3} \frac{\varepsilon}{2^s} \right.$$

We define $Q^b := \{([x_{n-1}, b], b)\}$ so that $S(f, Q^b, \alpha) = f(b)(b - x_{n-1})$ and therefore $|S(f, Q^b, \alpha)| \leq |f(b)|(b - x_{n-1}) \leq \frac{1}{3}\varepsilon$. Now

$$P = Q_1 \cup \cdots \cup Q_{s-1} \cup Q_s \cup Q^b$$

and so, by the Triangle Inequality,

$$\begin{aligned}
|S(f,P,\alpha) - A| &= |\sum_{i=1}^{s} S(f,Q_i,\alpha) + S(f,Q^b) - A| \\
&\leq |\sum_{i=1}^{s} S(f,Q_i,\alpha) - \int_a^{x_{n-1}} f d\alpha| + |S(f,Q^b,\alpha)| \\
&+ |\int_a^{x_{n-1}} f d\alpha - A| \\
&< \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon \\
&= \varepsilon
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $f \in GRS(I, \alpha)$ with integral A. □

We have only considered intervals of the form $I = [a, b]$ so far. Now we shall set up the theory for infinite intervals, that is, intervals of the form $I = [a, \infty), I = (-\infty, b]$ or $I = (-\infty, \infty)$. Most of the results from earlier chapters also hold for the infinite case with only minor adjustments. Of particular interest will be the application of Hake's Theorem in the proof of the Fundamental Theorem, which reflects the proof of this result in the case of the Generalised Riemann integral.

Definition 30. For $I = [a, \infty)$, a **tagged partition** of I is a collection of the form

$$\{([x_0, x_1], t_1), \ldots, ([x_{n-1}, x_n], t_n), ([x_n, \infty], \infty)\}$$

For $I = (\infty, b]$, a **tagged partition** of I is a collection of the form

$$\{([-\infty, x_1], -\infty), ([x_1, x_2], t_2), \ldots, ([x_n, x_{n+1}], t_{n+1})\}$$

For $I = (-\infty,)$, a **tagged partition** of I is a collection of the form

$$\{([-\infty, x_1], -\infty), ([x_1, x_2], t_2), \ldots, ([x_n, x_{n+1}], t_{n+1}), ([x_{n+1}, \infty], \infty)\}$$

We extend a function $f : [a, \infty) \to \mathbb{R}$ to $f : [a, \infty] \to \mathbb{R}$ by defining $f(\infty) := 0$. Similarly, we extend $g : (-\infty, b] \to \mathbb{R}$ to $g : [-\infty, b] \to \mathbb{R}$ by defining $g(-\infty) = 0$, and extend $h : (\infty, \infty) \to \mathbb{R}$ to $h : [-\infty, \infty] \to \mathbb{R}$ by defining $h(-\infty) = h(\infty) = 0$. If $\alpha : I \to \mathbb{R}$ is an integrator function, then it can be extended by setting $\alpha(-\infty) = \alpha(\infty) = 0$, although the choice of value at the endpoints is irrelevant.

In this way, the Riemann-Stieltjes sum as defined earlier makes sense as a sum of extended real numbers, using the convention that $\infty \times 0 = -\infty \times 0 = 0$.

Definition 31. A **gauge** on an infinite interval I is a strictly positive function on I.

Definition 32. A partition P of $[a, \infty]$ is δ-fine for a gauge $\delta : [a, \infty] \to (0, \infty)$ if the finite intervals satisfy

$$[x_{i-1}, x_i] \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)] \quad \text{for} \quad i = 1, \ldots, n$$

and the infinite interval satisfies

$$[x_n, \infty] \subseteq [\frac{1}{\delta(\infty)}, \infty]$$

A partition P of $[-\infty, b]$ is δ-fine for a gauge $\delta : [-\infty, b] \to (0, \infty)$ if the finite intervals satisfy

$$[x_{i-1}, x_i] \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)] \quad \text{for} \quad i = 2, \ldots, n+1$$

and the infinite interval satisfies

$$[-\infty, x_1] \subseteq [-\infty, -\frac{1}{\delta(-\infty)}]$$

A partition P of $[-\infty, \infty]$ is δ-fine for a gauge $\delta : [-\infty, \infty] \to (0, \infty)$ if the finite intervals satisfy

$$[x_{i-1}, x_i] \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)] \quad \text{for} \quad i = 2, \ldots, n+1$$

and the infinite intervals satisfy

$$[-\infty, x_1] \subseteq [-\infty, -\frac{1}{\delta(-\infty)}] \quad \text{and} \quad [x_n, \infty] \subseteq [\frac{1}{\delta(\infty)}, \infty]$$

Definition 33. Let I be an infinite interval. A function $f : I \to \mathbb{R}$ is Generalised Riemann Stieltjes integrable on I with respect to $\alpha : I \to \mathbb{R}$ if there exists $L \in \mathbb{R}$ such that for all $\varepsilon > 0$ there exists a gauge $\delta_\varepsilon : I \to (0, \infty)$ such that if P is a δ_ε-fine partition of I then

$$|S(f, P, \alpha) - L| < \varepsilon$$

Then L is the Generalised Riemann integral of f over I with respect to α, and we write

$$\int_I f d\alpha = L$$

Theorem 54. *(Existence of δ-fine partitions) If $\delta : [a, \infty]$ is a gauge then there exist δ-fine partitions of $[a, \infty]$.*

Proof. We choose $b \in \mathbb{R}$ such that

$$b \geq \max\{a, \frac{1}{\delta(\infty)}\}$$

The interval $[a, b]$ has a $\delta|[a, b]$-fine partition $P = \{(I_i, t_i)\}_{i=1}^n$ by Theorem 2. We define $I_{n+1} = [b, \infty]$ and $t_{n+1} = \infty$. The tagged partition $P \cup ([b, \infty], \infty)$ is a δ-fine partition of $[a, \infty]$. □

Similarly, it can be shown that if δ is a gauge on $[-\infty, b]$ or $[-\infty, \infty]$ then there exist δ-fine partitions of those intervals.

Theorem 55. *(Hake for $[a, \infty]$) Let $f, \alpha : [a, \infty] \to \mathbb{R}$ be functions. Then $f \in GRS([a, \infty], \alpha)$ if and only if $f \in GRS([a, c], \alpha)$ for all compact intervals $[a, c]$ with $c \in [a, \infty)$, and there exists $A \in \mathbb{R}$ such that*

$$\lim_{c \to \infty} \int_a^c f d\alpha = A$$

If so then

$$\int_a^\infty f d\alpha = A$$

We can prove generalisations of Hake's Theorem to $[-\infty, b]$ and $[-\infty, \infty]$ similarly.

Proof. For ($\Rightarrow$): Let $A = \int_a^\infty f d\alpha$. Fix $\varepsilon > 0$. Since $f \in GRS([a, \infty], \alpha)$, there exists a gauge γ on $[a, \infty]$ such that if P is a γ-fine partition then $|S(f, P, \alpha) - A| < \frac{\varepsilon}{2}$. Let x_n be the second last partition point, and let $c \geq x_n$ be finite. The Additivity Theorem carries over to the infinite interval case with no changes, so the function f is integrable on $[a, c]$. Hence, there exists a gauge γ_c on $[a, c]$ such that if P_c is a γ_c-fine partition of $[a, c]$ then

$$\left| S(f, P_c, \alpha) - \int_a^c f d\alpha \right| \leq \frac{\varepsilon}{2}$$

We assume that $\gamma_c(t) \leq \gamma(t)$ for all $t \in [a, c]$, as otherwise we could redefine γ_c to be the minimum of the two. We define the partition P_c^* as

$$P_c^* := P_c \cup ([c, \infty], \infty)$$

Then P_c^* is γ-fine since $c \geq x_n \geq \frac{1}{\delta(\infty)}$. We have

$$S(f, P_c^*, \alpha) = S(f, P_c, \alpha) + f(\infty) l([c, \infty)) = S(f, P_c, \alpha)$$

By the Triangle Inequality, we have

$$\begin{aligned} \left| \int_a^c f d\alpha - A \right| &\leq \left| \int_a^c f d\alpha - S(f, P_c^*, \alpha) \right| + |S(f, P_c^*, \alpha) - A| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

for any $c \geq x_n$. Since $\varepsilon > 0$ was arbitrary, we conclude that

$$\lim_{c \to \infty} \int_a^c f d\alpha = A$$

For ($\Leftarrow$): Suppose that there exists $A \in \mathbb{R}$ such that for all $c \in (a, \infty)$, the function $f|_{[a,c]}$ is integrable and

$$\lim_{c \to \infty} \int_a^c f d\alpha = A$$

Let $(c_k)_{k=0}^{\infty}$ be a strictly increasing sequence with

$$c_0 = a \quad \text{and} \quad \lim_{k \to \infty} c_k = \infty$$

Fix $\varepsilon > 0$. We choose $r \in \mathbb{N}$ such that

$$b \geq c_r \to \left| \int_a^b f d\alpha - A \right| < \frac{\varepsilon}{2}$$

If $k \in \mathbb{N}$ then let δ_k be a gauge on $I_k := [c_{k-1}, c_k]$ such that if a partition P_k is δ_k-fine then

$$\left| S(f, P_k, \alpha) - \int_{I_k} f d\alpha \right| < \frac{\varepsilon}{2^{k+1}}$$

Without loss of generality (by redefining if necessary), we assume that

$$\delta_1(c_0) \leq \frac{1}{2}(c_1 - c_0)$$

$$\text{For} \quad k \geq 1 : \delta_{k+1}(c_k) \leq \min\{\delta_k(c_k), \frac{1}{2}(c_k, \{c_{k-1}, c_{k+1}\})\}$$

$$\text{For} \quad k \geq 1 : \delta_k(t) \leq \frac{1}{2}(t, \{c_{k-1}, c_k\}) \quad \text{for} \quad t \in (c_{k-1}, c_k)$$

We define a gauge $\delta : [a, \infty] \to (0, \infty)$ by

$$\delta(t) = \begin{cases} \delta_k(t) & \text{if} \quad t \in I_k, k \in \mathbb{N} \\ \frac{1}{c_r} & \text{if} \quad t = \infty \end{cases}$$

Let P be a δ-fine partition of $[a, \infty]$. By δ-fineness, we have $c_r \leq x_n$. Let $s \in \mathbb{N}$ be the smallest positive integer such that $x_n \leq c_s$. So we must have $r \leq s$. If $k \in \{1, \ldots, s-1\}$ then c_k must be the tag for any subinterval in the partition containing it. By splitting such intervals in two at c_k, we may assume that $c_0, \ldots, c_{s-1}$ are some of the partition points. We define

$$Q_1 := P \cap [c_0, c_1], \ldots, Q_{s-1} := P \cap [c_{s-2}, c_{s-1}], Q_s := P \cap [c_{s-1}, x_n]$$

Now, each Q_k, for $k \in \{1, \ldots, s-1\}$, is a δ_k-fine partition of I_k and so

$$\left| S(f, Q_k, \alpha) - \int_{I_k} f d\alpha \right| < \frac{\varepsilon}{2^{k+1}}$$

Also, Q_s is a δ_s-fine subpartition. We may apply the Saks-Henstock Lemma to infinite intervals to conclude that

$$\left| S(f, Q_s, \alpha) - \int_{c_{s-1}}^{x_n} f d\alpha \right| < \frac{\varepsilon}{2^{s+1}}$$

If we define $Q_\infty := \{([x_n, \infty], \infty\}$ then $S(f, Q_\infty, \alpha) = 0$. Now

$$P = Q_1 \cup \cdots \cup Q_s \cup Q_\infty$$

and so, by the Triangle Inequality,

$$\begin{aligned}
|S(f, P, \alpha) - A| &= \left| \sum_{i=1}^{s} S(f, Q_i, \alpha) + S(f, Q_\infty, \alpha) - A \right| \\
&\leq \left| \sum_{i=1}^{s} S(f, Q_i, \alpha) - \int_a^{x_n} f d\alpha \right| + |S(f, Q_\infty, \alpha)| \\
&+ \left| \int_a^{x_n} f d\alpha - A \right| \\
&< \frac{\varepsilon}{2} + 0 + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $f \in GRS([a, \infty], \alpha)$ with integral A. □

Theorem 56. *(Fundamental Theorem) Suppose that* $\alpha : [a, \infty) \to \mathbb{R}$ *is continuous and strictly increasing. Suppose that* $f, F : [a, \infty) \to \mathbb{R}$ *are such that*

- *The function F is continuous on $[a, \infty)$ and* $\lim_{x\to\infty} F(x)$.
- *We have $D_\alpha F(x) = f(x)$ for all $x \in (a, \infty)$.*

Then $f \in GRS([a, \infty), \alpha)$ and

$$\int_a^\infty f d\alpha = \lim_{x\to\infty} [F(x) - F(a)]$$

We can prove generalisations of the Fundamental Theorem to $(-\infty, b]$ and $[-\infty, \infty]$ similarly.

Proof. If $\gamma \in (a, \infty)$ then we apply the Fundamental Theorem for closed, bounded intervals to $[a, \gamma]$ to conclude that $f \in GRS([a, \gamma], \alpha)$ and

$$\int_a^\gamma f d\alpha = F(\gamma) - F(a)$$

Now we let $\gamma \to \infty$ and apply Hake's Theorem to conclude that $f \in GRS([a, \infty), \alpha)$ and

$$\int_a^\infty f d\alpha = \lim_{x\to\infty} [F(x) - F(a)]$$

□

Chapter 10

Reduction to Finite Sums

In this section, we consider some original applications for Generalised Riemann-Stieltjes integrals involving a particular choice of integrator function α - step functions. In doing so, we prove results which convert an integral into a finite sum based on similar theorems involving the Riemann-Stieltjes integral.

The following theorems are based on well-known properties of the Riemann-Stieltjes integral given in [3] and [4].

Theorem 57. *Suppose that $a < c < b$ and that the function $\alpha : I \to \mathbb{R}$ is given by*

$$\alpha(x) = \begin{cases} \alpha(a) & \text{if} \quad a \leq x < c \\ k & \text{if} \quad x = c \\ \alpha(b) & \text{if} \quad c < x \leq b \end{cases}$$

where $\alpha(a), k, \alpha(b) \in \mathbb{R}$. Then all $f : I \to \mathbb{R}$ are integrable on I with respect to α and

$$\int_a^b f d\alpha = f(c)[\alpha(c^+) - \alpha(c^-)]$$

Proof. We choose the gauge $\delta : I \to (0, \infty)$ given by

$$\delta(x) = \begin{cases} \frac{1}{2}(x, c) & \text{if} \quad x \neq c \\ 1 & \text{if} \quad x = c \end{cases}$$

Then, for a δ-fine partition P, any subinterval containing c must have c as its tag. If c is not one of the partition points, then we calculate the Riemann-Stieltjes sum

$$\begin{aligned} S(f, P, \alpha) &= \sum_{i=1}^{n} f(t_i)[\alpha(x_i) - \alpha(x_{i-1})] \\ &= f(c)[\alpha(x_j) - \alpha(x_{j-1})] \quad \text{where } [x_{j-1}, x_j] \text{ has } c \text{ as its tag} \\ &= f(c)[\alpha(c^+) - \alpha(c^-)] \end{aligned}$$

If c is one of the partition points, then it is the tag for precisely two intervals, and we calculate the Riemann sum

$$\begin{aligned} S(f, P, \alpha) &= \sum_{i=1}^{n} f(t_i)[\alpha(x_i) - \alpha(x_{i-1})] \\ &= f(c)[\alpha(c) - \alpha(x_{j-1})] + f(c)[\alpha(x_{j+1}) - \alpha(c)] \\ &= f(c)[\alpha(c^+) - \alpha(c^-)] \\ &= f(c)[\alpha(c^+) - \alpha(c^-)] \end{aligned}$$

So, in both cases, we have $|S(f, P, \alpha) - f(c)[\alpha(c^+) - \alpha(c^-)]| = 0$, and therefore $f \in GRS(I, \alpha)$ and

$$\int_a^b f d\alpha = f(c)[\alpha(c^+) - \alpha(c^-)]$$

□

Definition 34. Suppose $\alpha : I \to \mathbb{R}$ is a step function. Then the number $\alpha(x_{k^+}) - \alpha(x_{k^-})$ is called the **jump** at x_k, for $1 < k < n$. The jump at x_1 is given by $\alpha(x_{1^+}) - \alpha(x_1)$, and the jump at x_n is given by $\alpha(x_n) - \alpha(x_{n^-})$

Theorem 58. *Let $\alpha : I \to \mathbb{R}$ be a step function with jump α_k at x_k. Then all $f : I \to \mathbb{R}$ are integrable on I with respect to α, and*

$$\int_a^b f(x)d\alpha(x) = \sum_{k=1}^{n} f(x_k)\alpha_k$$

Proof. We have, using the Additivity Theorem and the previous theorem,

$$\int_a^b f(x)d\alpha(x) = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(x)d\alpha(x) = \sum_{k=1}^{n} f(x_k)\alpha(x_k)$$

□

Definition 35. We define the **greatest integer function** $\lfloor x \rfloor : \mathbb{R} \to \mathbb{R}$ by choosing $\lfloor x \rfloor$ to be the greatest integer less than or equal to x.

Remark. Note that $\lfloor x \rfloor$ is the unique number satisfying

$$\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$$

Theorem 59. *Every finite sum can be written as a Generalised Riemann-Stieltjes integral.*

Proof. Let $\sum_{k=1}^{n} a_k$ be a finite sum. Define $f : [0, n] \to \mathbb{R}$ by

$$f(x) = \begin{cases} a_k & \text{if} \quad k-1 < x \leq k \\ 0 & \text{if} \quad x = 0 \end{cases}$$

The greatest integer function is a step function with a jump of 1 at each integer. Applying the previous theorem, we have

$$\int_0^n f(x)d\lfloor x \rfloor = \sum_{k=1}^{n} f(k) = \sum_{k=1}^{n} a_k$$

□

Remark. The Riemann-Liouville fractional integral (which we will see later in more detail) can be approximated as a Generalised Riemann-Stieltjes integral and therefore as a finite sum. We use this process to define a discrete approximation to this integral in the following original work:

We have

$$D_0^{-\mu} f(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-u)^{\mu-1} f(u)du = \int_0^t f(u)d\alpha(u)$$

where $\alpha : [0,t] \to \mathbb{R}$ is given by the primitive of $(t-u)^{\mu-1}/\Gamma(\mu)$, which is

$$\alpha(x) = -\frac{(t-u)^{\mu}}{\Gamma(\mu+1)}$$

We partition the interval $[0,t]$ with $\lfloor t \rfloor$ equally spaced partition points u_i given by

$$u_i = \frac{it}{\lfloor t \rfloor} \quad \text{for } i = 1, \ldots, \lfloor t \rfloor$$

We then approximate the function α by $\widetilde{\alpha} : I \to \mathbb{R}$, given by

$$\widetilde{\alpha}(u) = \alpha(u_i) \quad \text{for} \quad u_{i-1} \le u < u_i$$

Now, since $\widetilde{\alpha}$ is a step function, we can use Theorem 58 to write the integral $\int_0^t f(u)d\widetilde{\alpha}(u)$ as a finite sum. We note that for $i = 1, \ldots, \lfloor t \rfloor - 1$ we have a jump of $\alpha(u_{i+1}) - \alpha(u_i)$ at the partition point u_i, and for $i = \lfloor t \rfloor$ we have a jump of 0. So, we obtain

$$\int_0^t f(u)d\alpha(u) \approx \int_0^t f(u)d\widetilde{\alpha}(u) = \sum_{i=1}^{\lfloor t \rfloor - 1} f(u_i)[\alpha(u_{i+1}) - \alpha(u_i)]$$

As a consequence of the results on the conversion of Riemann-Stieltjes integrals to finite sums, we are able to rewrite two common integral transforms in another way in the following original application:

Theorem 60. *It is possible to write both the Laplace transform and Z-transform in the form of a Generalised Riemann-Stieltjes integral*

$$\int_0^\infty f(x)d\alpha_\varepsilon(x)$$

Proof. For the Laplace transform, we define

$$\alpha_\varepsilon(x) = -\frac{e^{-\varepsilon x}}{\varepsilon}$$

for $\varepsilon > 0$. We have

$$\begin{aligned}\int_0^\infty f(x)d\alpha_\varepsilon(x) &= \int_0^\infty f(x)\alpha_\varepsilon'(x)dx \\ &= \int_0^\infty f(x)e^{-\varepsilon x}dx\end{aligned}$$

which is the Laplace transform. For the Z-transform, we define

$$\alpha_\varepsilon(x) = \frac{1}{1-\frac{1}{\varepsilon}}\sum_{j=1}^{\infty} \mathbb{1}_{[j-1,j)}(x)(1-\varepsilon^{-j})$$

for $\varepsilon > 1$. Now, we can write

$$\alpha_\varepsilon(x) = \frac{1}{1-\frac{1}{\varepsilon}}[\mathbb{1}_{[0,1)}(x)(1-\varepsilon^{-1}) + \mathbb{1}_{[1,2)}(x)(1-\varepsilon^{-2}) + \cdots]$$

or, equivalently,

$$\alpha_\varepsilon(x) = \begin{cases} 1 & \text{if} \quad 0 \le x < 1 \\ \vdots & \\ 1 + \varepsilon^{-1} + \cdots + \varepsilon^{-(k-1)} & \text{if} \quad k-1 \le x < k \\ \vdots & \end{cases}$$

We now consider the interval $[0, n]$. We have

$$\int_0^n f(x)d\alpha_\varepsilon(x) = \sum_{k=1}^{n} f(k)[\alpha_\varepsilon(x_{k^+}) - \alpha_\varepsilon(x_{k^-})] = \sum_{k=1}^{n} f(k)\varepsilon^{-k}$$

So on the interval $[0, \infty]$ we have

$$\int_0^\infty f(x)d\alpha_\varepsilon(x) = \sum_{k=1}^{\infty} f(k)\varepsilon^{-k}$$

This is a Z-transform starting at $k = 1$ (not $k = 0$). □

Chapter 11

Fractional Calculus

The process of defining higher order integer derivatives is very natural - simply taking iterated derivates an integer number of times. Fractional calculus is motivated by the question of whether it is possible to define fractional (or even real or complex) order derivatives. Unfortunately, unlike with higher order integer derivates, there is no obvious choice of definition, and many approaches have been tried since the question was first posed in the 1700s. Ideas include interpolation between the appropriate integer derivatives, and modifying the formulas for integer derivatives using, for example, the gamma function. However, today's definitions are unusual in that first a fractional integral is defined, and then that is used to create the fractional derivative. The reason for this is that both the Riemann-Liouville and Caputo fractional derivates operate on the same principle - that integration and differentiation are natural "inverses" of each other (for example, in the Fundamental Theorem, we see that the integral of a derivative is the original function). The fractional derivatives use the idea that by taking an integer number of ordinary derivatives, and a fractional number of integrals, the two will "cancel each other out" in order to obtain a fractional number of derivatives.

Suppose $T > 0$. In this chapter, f is a real-valued function on $[0, T]$.

Definition 36. Suppose $\gamma \in \mathbb{R}_{>0}$. We define the **Riemann-Liouville fractional integral** of a function f to be the integral

$$D_t^{-\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-u)^{\gamma-1} f(u) du$$

Example 19. Let $f(x) = x^k$ for some $k > 0$. We calculate the Riemann-Liouville fractional integral of f: We have

$$\begin{aligned} D_x^{-\gamma}(x^k) &= \frac{1}{\Gamma(\gamma)} \int_0^x (x-u)^{\gamma-1} u^k du \\ &= \frac{1}{\Gamma(\gamma)} \int_0^x (1-\frac{u}{x})^{\gamma-1} x^{\gamma-1} u^k du \end{aligned}$$

Now we perform a substitution, $v = \frac{u}{x}$, and hence $du = xdv$:

$$\begin{aligned} D_x^{-\gamma}(x^k) &= \frac{1}{\Gamma(\gamma)} \int_0^1 (1-v)^{\gamma-1} x^{\gamma-1} (xv)^k x dv \\ &= \frac{1}{\Gamma(\gamma)} x^{\gamma+k} \int_0^1 v^k (1-v)^{\gamma-1} dv \\ &= \frac{1}{\Gamma(\gamma)} x^{\gamma+k} B(k+1, \gamma) \\ &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+k+1)} x^{\gamma+k} \end{aligned}$$

where B represents the beta function.

We then use the integral to define the fractional derivative. There are two possible ways of doing this:

Definition 37. Suppose $\gamma \in \mathbb{R}_{>0}$. Let n be the least integer greater than or equal to γ. The **Riemann-Liouville fractional derivative** of a function f is given by

$$D^\gamma f(t) = \frac{d^n}{dt^n} D^{(-n-\gamma)} f(t) = \frac{d^n}{dt^n} \Big[\frac{1}{\Gamma(n-\gamma)} \int_0^t (t-u)^{n-\gamma-1} f(u) du \Big]$$

Definition 38. Suppose $\gamma \in \mathbb{R}_{>0}$. Let n be the least integer greater than or equal to γ. The **Caputo fractional derivative** of a function f is given by

$$D^{\gamma} f(t) = D^{-(n-\gamma)} \frac{d^n}{dt^n} f(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t (t-u)^{n-\gamma-1} f^{(n)}(u) du$$

In the Riemann-Liouville fractional derivative, the integration is performed first, and then the differentiation, whereas in the Caputo fractional derivative, the operations are performed in the reverse order. These will not necessarily give the same answer, as we can see in the example below.

Remark. Note that the Caputo fractional derivative of a constant is 0, reflecting the behaviour of the ordinary derivative, while the Riemann-Liouville derivative does not share this property.

Example 20. Continuing Example 19, we will calculate both the Riemann-Liouville and Caputo fractional derivatives. Suppose $\gamma = 0.5$, and so $n = 1$. Firstly, we find the Riemann-Liouville fractional derivative:

$$\begin{aligned} D^{0.5}(x^k) &= \frac{d}{dx} D^{-(1-0.5)} f(x) \\ &= \frac{d}{dx} D^{-0.5} f(x) \\ &= \frac{\Gamma(1.5)}{\Gamma(1.5+k)} x^{k+0.5} \end{aligned}$$

Next, we find the Caputo fractional derivative. We have $f^{(1)}(u) = \frac{d}{du}(u^k) = ku^{(k-1)}$. Now

$$\begin{aligned} D^{0.5}(x^k) &= \frac{1}{\Gamma(1-0.5)} \int_0^x (x-u)^{1-0.5-1} k u^{(k-1)} du \\ &= \frac{k}{\Gamma(0.5)} \int_0^x (x-u)^{-0.5} u^{(k-1)} du \\ &= \frac{k}{\Gamma(0.5)} \int_0^x (1-\frac{u}{x})^{-0.5} u^{(k-1)} du \end{aligned}$$

We substitute $v = \frac{u}{x}$ so that $xdv = du$, and therefore

$$\begin{aligned} D^{0.5}(x^k) &= \frac{k}{\Gamma(0.5)} \int_0^1 (1-v)^{-0.5}(vx)^{(k-1)} dv \\ &= \frac{kx^{k-1}}{\Gamma(0.5)} \int_0^1 v^{(k-1)}(1-v)^{-0.5} dv \\ &= \frac{kx^{k-1}}{\Gamma(0.5)} B(k, 0.5) \\ &= \frac{kx^{k-1}\Gamma(k)}{\Gamma(k+0.5)} \end{aligned}$$

Chapter 12

Generalised Fractional Calculus

In this chapter, we generalise the three main definitions from fractional calculus that we saw in the previous section using the α-derivative and Generalised Riemann-Stieltjes integral. What were previously ordinary derivatives will be changed to α-derivatives, and what were previously Riemann (or Lebesgue) integrals will be changed to Generalised Riemann-Stieltjes integrals (or "α-integrals"). The motivation behind the creation of these new definitions is that firstly, we can prove existence of the integrals and derivatives using the existence results from Chapter 8, secondly, we may be able to use Hake's Theorem to extend the definitions to infinite intervals, and finally, the Generalised Riemann-Stieltjes integral can integrate a wider range of functions than the other integrals we've seen. The work in this chapter is entirely original.

We begin with an analogue of the Riemann-Liouville fractional integral.

Definition 39. Suppose $T > 0, \mu > 1 \in \mathbb{R}$, and α, f are real-valued functions on $[0, T]$. The **Riemann-Liouville fractional α-integral** is given by

$$D_\alpha^{-\mu} f(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-u)^{\mu-1} f(u) d\alpha(u)$$

Theorem 61. *If α is Lipschitz continuous and increasing, and f is either regulated or bounded below on $[0,t]$ then $D_\alpha^{-\mu} f(t)$ exists.*

Proof. Suppose that f is regulated. We know that $(t-u)^{\mu-1}$ is continuous on $[0,t]$, and therefore bounded on $[0,t]$ (a continuous function on a compact interval), and therefore bounded below on $[0,t]$. So, using Theorem 50, we conclude that the integral exists. Suppose that f is bounded below on $[0,t]$. We know that $(t-u)^{\mu-1}$ is continuous on $[0,t]$, and therefore uniformly continuous and hence regulated on $[0,t]$. So by Theorem 50, we conclude that the integral exists. □

Example 21. We define $\alpha : [0,1] \to \mathbb{R}$ by

$$\alpha(x) = \begin{cases} x & \text{if} \quad 0 \leq x \leq 0.5 \\ x^2 + 0.5 & \text{if} \quad 0.5 < x \leq 1 \end{cases}$$

and $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if} \quad 0 \leq x \leq 0.25 \\ 1 & \text{if} \quad 0.25 < x \leq 1 \end{cases}$$

Now, α is strictly increasing, and is piecewise smooth with bounded derivative, hence Lipschitz. But α is not $C^1([0,1])$ so we may not use the Simplification Theorem to convert to the ordinary case. But, using Theorem 61, we know that the Riemann-Liouville fractional α-integral of f exists. This example shows that the new definitions do not simply reduce to the familiar ones from fractional calculus, but have their own independent merit.

Next we see an analogue of the Riemann-Liouville fractional derivative. As in the previous section on fractional calculus, we will use the fractional integral that we just saw to define both of the derivatives.

Definition 40. Suppose $T > 0, \mu > 1 \in \mathbb{R}$, and α, f are real-valued functions on $[0, T]$. The **Riemann-Liouville fractional α-derivative** is given by

$$\begin{aligned} RL(f, \alpha, \mu) &= D_\alpha^{\lceil \mu \rceil} D_\alpha^{-(\lceil \mu \rceil - \mu)} f(t) \\ &= D_\alpha^{\lceil \mu \rceil} \left[\frac{1}{\Gamma(\lceil \mu \rceil - \mu)} \int_0^t (t-u)^{\lceil \mu \rceil - \mu - 1} f(u) d\alpha(u) \right] \end{aligned}$$

Theorem 62. *If $f : [0, t] \to \mathbb{R}$ is analytic and $\alpha'(u) = Mu^k$ or $\alpha'(u) = M(t-u)^k$ for some $k, M \in \mathbb{R}$ then $RL(f, \alpha, \mu)$ exists.*

Proof. Since f is analytic, we have

$$f(u) = \sum_{n=0}^{\infty} a_n u^n$$

and so

$$\begin{aligned} &\frac{1}{\Gamma(\lceil \mu \rceil - \mu)} \int_0^t (t-u)^{\lceil \mu \rceil - \mu - 1} f(u) d\alpha(u) \\ =&\frac{1}{\Gamma(\lceil \mu \rceil - \mu)} \sum_{n=0}^{\infty} a_n \int_0^t (t-u)^{\lceil \mu \rceil - \mu - 1} u^n d\alpha(u) \end{aligned}$$

We now apply the Simplification Theorem:

$$RL(f, \alpha, \mu) = \frac{1}{\Gamma(\lceil \mu \rceil - \mu)} \sum_{n=0}^{\infty} a_n \int_0^t (t-u)^{\lceil \mu \rceil - \mu - 1} u^n \alpha'(u) du$$

If $\alpha'(u) = Mu^k$ then we have, applying a substitution $v = \frac{u}{t}$,

$$
\begin{aligned}
RL(f,\alpha,\mu) &= \frac{M}{\Gamma(\lceil\mu\rceil-\mu)}\sum_{n=0}^{\infty} a_n \int_0^t (t-u)^{\lceil\mu\rceil-\mu-1} u^{n+k} du \\
=&\frac{M}{\Gamma(\lceil\mu\rceil-\mu)}\sum_{n=0}^{\infty} a_n \int_0^t (1-\frac{u}{t})^{\lceil\mu\rceil-\mu-1} t^{\lceil\mu\rceil-\mu-1} \left(\frac{u}{t}\right)^{n+k} t^{n+k} du \\
=&\frac{M}{\Gamma(\lceil\mu\rceil-\mu)}\sum_{n=0}^{\infty} a_n t^{\lceil\mu\rceil-\mu+n+k-1} \int_0^1 (1-v)^{\lceil\mu\rceil-\mu-1} v^{n+k} t dv \\
=&\frac{M}{\Gamma(\lceil\mu\rceil-\mu)}\sum_{n=0}^{\infty} a_n B(n+k+1,\lceil\mu\rceil-\mu) t^{\lceil\mu\rceil-\mu+n+k} \\
=&\frac{M}{\Gamma(\lceil\mu\rceil-\mu)}\sum_{n=0}^{\infty} a_n \frac{\Gamma(n+k+1)\Gamma(\lceil\mu\rceil-\mu)}{\Gamma(n+k+\lceil\mu\rceil-\mu+1)} t^{\lceil\mu\rceil-\mu+n+k} \\
=&M\sum_{n=0}^{\infty} a_n \frac{\Gamma(n+k+1)}{\Gamma(n+k+\lceil\mu\rceil-\mu+1)}
\end{aligned}
$$

If $\alpha'(u) = M(t-u)^k$ then we have, applying a substitution $v = \frac{u}{t}$,

$$
\begin{aligned}
RL(f,\alpha,\mu) &= \frac{M}{\Gamma(\lceil\mu\rceil-\mu)}\sum_{n=0}^{\infty} a_n \int_0^t (t-u)^{\lceil\mu\rceil-\mu-1} u^n (t-u)^k du \\
&= \frac{M}{\Gamma(\lceil\mu\rceil-\mu)}\sum_{n=0}^{\infty} a_n \int_0^t (t-u)^{\lceil\mu\rceil-\mu+k-1} u^n du \\
&= \frac{M}{\Gamma(\lceil\mu\rceil-\mu)}\sum_{n=0}^{\infty} a_n \int_0^t (1-\frac{u}{t})^{\lceil\mu\rceil-\mu+k-1} t^{\lceil\mu\rceil-\mu+k-1} \left(\frac{u}{t}\right)^n t^n du \\
&= \frac{M}{\Gamma(\lceil\mu\rceil-\mu)}\sum_{n=0}^{\infty} a_n \int_0^1 (1-v)^{\lceil\mu\rceil-\mu+k-1} v^n t^{\lceil\mu\rceil-\mu+k+n-1} t dv \\
&= \frac{M}{\Gamma(\lceil\mu\rceil-\mu)}\sum_{n=0}^{\infty} a_n B(n+1,\lceil\mu\rceil-\mu+k) t^{\lceil\mu\rceil-\mu+k+n}
\end{aligned}
$$

where B represents the beta function. □

Finally, we define an analogue to the Caputo fractional derivative.

Definition 41. Suppose $T > 0$, f and α are real-valued functions on $[0,T]$, and $\mu > 1 \in \mathbb{R}$. We define, for $t \in [0,T]$, the **Caputo fractional α-**

derivative

$$\begin{aligned} C(f,\alpha,\mu)(t) &= D_\alpha^{-(\lceil\mu\rceil-\mu)} D_\alpha^{\lceil\mu\rceil} f(t) \\ &= \frac{1}{\Gamma(\lceil\mu\rceil-\mu)} \int_0^t (t-u)^{\lceil\mu\rceil-\mu-1} [D_\alpha^{\lceil\mu\rceil} f(u)] d\alpha(u) \end{aligned}$$

Remark. Note that for $\alpha(x) = x$, the above definition reduces to the ordinary Caputo derivative that we saw in the previous chapter.

Theorem 63. *If α is increasing and Lipschitz continuous, and one of the following is true:*

- *The function f is $\lceil\mu\rceil$-times continuously α-differentiable on $[0,t]$.*
- *The function f is $(\lceil\mu\rceil-1)$-times continuously α-differentiable on $[0,t]$ and $D_\alpha^{\lceil\mu\rceil} f$ exists and is bounded below on $[0,t]$.*

Then the Caputo fractional α-derivative of f exists.

Proof. Case 1: Suppose f is $\lceil\mu\rceil$-times continuously α-differentiable on $[0,t]$. Firstly, the function $h(u) := (t-u)^{\lceil\mu\rceil-\mu-1}$ is bounded below because h is a continuous function on a compact interval, and is therefore bounded. Secondly, the function $D_\alpha^{\lceil\mu\rceil} f(u)$ is regulated. Since f is $\lceil\mu\rceil$-times continuously differentiable, we know that $D_\alpha^{\lceil\mu\rceil} f(u)$ is continuous. Now, a continuous function on a closed interval is uniformly continuous, and therefore it is regulated. By Theorem 50, we conclude that the integral exists.

Case 2: Suppose f is $(\lceil\mu\rceil-1)$-times continuously α-differentiable on $[0,t]$ and $D_\alpha^{\lceil\mu\rceil} f$ is bounded below on $[0,t]$ Note that the function $h(u) := (t-u)^{\lceil\mu\rceil-\mu-1}$ is continuous on $[0,t]$, therefore uniformly continuous on $[0,t]$, and hence regulated. Using Theorem 50, we conclude that the integral exists. □

Example 22. Suppose that $\mu > 1$, α is Lipschitz continuous and increasing, and $f(u) = k$ for all $u \in [0, T]$. We have

$$\begin{aligned} C(f, \alpha, \mu) &= \frac{1}{\Gamma(\lceil \mu \rceil - \mu)} \int_0^t (t-u)^{\lceil \mu \rceil - \mu - 1} D_\alpha^{\lceil \mu \rceil}(k) d\alpha(u) \\ &= \frac{1}{\Gamma(\lceil \mu \rceil - \mu)} \int_0^t (t-u)^{\lceil \mu \rceil - \mu - 1} \cdot 0 d\alpha(u) \\ &= 0 \end{aligned}$$

Note that this reflects the behaviour of the ordinary Caputo derivative of a constant.

Example 23. Suppose α is $C^1([0, T])$ and increasing, $\mu > 1$ and $f(u) = e^{\alpha(u)}$ is $\lceil \mu \rceil$-times α-differentiable on $[0, T]$. Note that $D_\alpha f(u) = \frac{\alpha'(u)e^{\alpha(u)}}{\alpha'(u)} = f(u)$. We have

$$\begin{aligned} C(f, \alpha, \mu) &= \frac{1}{\Gamma(\lceil \mu \rceil - \mu)} \int_0^t (t-u)^{\lceil \mu \rceil - \mu - 1} D_\alpha^{\lceil \mu \rceil} f(u) d\alpha(u) \\ &= \frac{1}{\Gamma(\lceil \mu \rceil - \mu)} \int_0^t (t-u)^{\lceil \mu \rceil - \mu - 1} \cdot e^{\alpha(u)} d\alpha(u) \\ &= \frac{1}{\Gamma(\lceil \mu \rceil - \mu)} \int_0^t (t-u)^{\lceil \mu \rceil - \mu - 1} \cdot e^{\alpha(u)} \alpha'(u) du \end{aligned}$$

using the Simplification Theorem. Note that $(t-u)^{\lceil \mu \rceil - \mu - 1} e^{\alpha(u)}$ is regulated since it is continuous, and therefore it is also Generalised Riemann-Stieltjes integrable with respect to α on $[0, T]$ by Theorem 46. Hence, it is appropriate to use the Simplification Theorem here. So

$$C(f, \alpha, \mu) = \frac{1}{\Gamma(\lceil \mu \rceil - \mu)} \int_0^t (t-u)^{\lceil \mu \rceil - \mu - 1} \frac{d}{du}(e^{\alpha(u)}) du$$

which is the ordinary Caputo derivative of $f(u) = e^{\alpha(u)}$.

Example 24. Suppose $f(u) = u^p, \alpha(u) = u^q$ where $p, q \geq 1$. We have

$$\begin{aligned} D_\alpha f(u) &= \frac{f'(u)}{\alpha'(u)} \\ &= \frac{pu^{p-1}}{qu^{q-1}} \\ &= \frac{p}{q} u^{p-q} \end{aligned}$$

and

$$\begin{aligned} D_\alpha^2 f(u) &= \frac{1}{qu^{q-1}}\frac{p}{q}(p-q)u^{p-q-1} \\ &= \frac{p(p-q)}{q^2}u^{p-2q} \end{aligned}$$

$$\begin{aligned} D_\alpha^3 f(u) &= \frac{p(p-q)}{q^2}\frac{1}{qu^{q-1}}(p-2q)u^{p-2q-1} \\ &= \frac{p(p-q)(p-2q)}{q^3}u^{p-3q} \end{aligned}$$

In general, we have

$$D_\alpha^n f(u) = \frac{p(p-q)(p-2q)\cdots(p-(n-1)q)}{q^n}u^{p-nq}$$

So

$$\begin{aligned} &C(f,\alpha,\mu) \\ &= \frac{1}{\Gamma(\lceil\mu\rceil-\mu)}\int_0^t (t-u)^{\lceil\mu\rceil-\mu-1}[D_\alpha^{\lceil\mu\rceil}f(u)]d\alpha(u) \\ &= \frac{1}{\Gamma(\lceil\mu\rceil-\mu)}\int_0^t (t-u)^{\lceil\mu\rceil-\mu-1}\frac{p(p-q)\cdots(p-(\lceil\mu\rceil-1)q)}{q^{\lceil\mu\rceil}}u^{p-\lceil\mu\rceil q}\alpha'(u)du \\ &= \frac{p(p-q)\cdots(p-(\lceil\mu\rceil-1)q)}{\Gamma(\lceil\mu\rceil-\mu)q^{\lceil\mu\rceil}}\int_0^t (t-u)^{\lceil\mu\rceil-\mu-1}u^{p-\lceil\mu\rceil q}qu^{q-1}du \\ &= \frac{p(p-q)\cdots(p-(\lceil\mu\rceil-1)q)}{\Gamma(\lceil\mu\rceil-\mu)q^{\lceil\mu\rceil-1}}\int_0^t (t-u)^{\lceil\mu\rceil-\mu-1}u^{p+q(1-\lceil\mu\rceil)-1}du \end{aligned}$$

using the Simplification Theorem. Let

$$K := \frac{p(p-q)\cdots(p-(\lceil\mu\rceil-1)q)}{\Gamma(\lceil\mu\rceil-\mu)q^{\lceil\mu\rceil-1}}$$

Now, we will perform a substitution $v = \frac{u}{t}$ in order to evaluate the integral:

$$\begin{aligned} &C(f,\alpha,\mu) \\ &= K\int_0^t (1-\frac{u}{t})^{\lceil\mu\rceil-\mu-1}t^{\lceil\mu\rceil-\mu-1}\left(\frac{u}{t}\right)^{p+q(1-\lceil\mu\rceil)-1}t^{p+q(1-\lceil\mu\rceil)-1}du \\ &= Kt^{\lceil\mu\rceil-\mu+p+q(1-\lceil\mu\rceil)-2}\int_0^1 (1-v)^{\lceil\mu\rceil-\mu-1}v^{p+q(1-\lceil\mu\rceil)-1}tdv \\ &= Kt^{\lceil\mu\rceil-\mu+p+q(1-\lceil\mu\rceil)-1}\int_0^1 (1-v)^{\lceil\mu\rceil-\mu-1}v^{p+q(1-\lceil\mu\rceil)-1}dv \\ &= Kt^{\lceil\mu\rceil-\mu+p+q(1-\lceil\mu\rceil)-1}B(p+q(1-\lceil\mu\rceil),\lceil\mu\rceil-\mu) \end{aligned}$$

where B represents the beta function. Now, we can simplify further:

$$
\begin{aligned}
&C(f,\alpha,\mu)\\
&= \frac{p(p-q)\cdots(p-(\lceil\mu\rceil-1)q)}{\Gamma(\lceil\mu\rceil-\mu)q^{\lceil\mu\rceil-1}} t^{\lceil\mu\rceil-\mu+p+q(1-\lceil\mu\rceil)-1} \frac{\Gamma(p+q(1-\lceil\mu\rceil))\Gamma(\lceil\mu\rceil-\mu)}{\Gamma(p+q(1-\lceil\mu\rceil)+\lceil\mu\rceil-\mu)}\\
&= \frac{p(p-q)\cdots(p-(\lceil\mu\rceil-1)q)}{q^{\lceil\mu\rceil-1}} \frac{\Gamma(p+q(1-\lceil\mu\rceil))}{\Gamma(p+q(1-\lceil\mu\rceil)+\lceil\mu\rceil-\mu)} t^{\lceil\mu\rceil-\mu+p+q(1-\lceil\mu\rceil)-1}
\end{aligned}
$$

Chapter 13

Conclusion

In summary, we have reviewed four types of integral (the Riemann integral and three related types), with particular focus on the Generalised Riemann-Stieltjes integral and its properties. We then presented an overview of the α-derivative and expanded its theory, as well as proving the Fundamental Theorem of Calculus for Generalised Riemann-Stieltjes integrals. Finally, we examined definitions and some basic examples from the field of fractional calculus, then applied the earlier work on the Generalised Riemann-Stieltjes integral to increase the scope of those definitions.

There are many more avenues of study to be considered based on the results we have seen so far. Naturally, it will be of interest to prove further existence results for the Riemann-Liouville fractional α-integral and α-derivative, as well as the Caputo fractional α-derivative. In particular, the Riemann-Liouville derivative can only be proven to exist under certain, specific conditions, and in further work it would be beneficial to expand the class of functions for which this definition exists.

Although we briefly considered the geometric interpretation of the Generalised Riemann-Stieltjes integral, further work is needed on the visualisation of this integral as the current results only apply under specific conditions on the integrand and integrator. As a consequence of the lack of suitable geometric meaning of the Riemann-Stieltjes integral, the definitions from fractional calculus are also difficult to understand geometrically, although [7] presents ideas about the interpretation of both the Riemann-Stieltjes integral as well as various integrals in fractional calculus. Further work should focus on the visualisation of the definitions in Chapter 12.

We examined a basic type of α-differential equation in Chapter 4, but this case simplified to an easy ordinary differential equation. There is scope for further exploration into the concepts of higher order α-derivatives, partial α-derivatives, α-differential equations and fractional α-differential equations. Future work would attempt to generalise the results given in [9].

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