COMPACTNESS THEOREMS IN RIEMANNIAN GEOMETRY

PENELOPE DRASTIK

1. Introduction

In this report, we discuss the concept of convergence of metric spaces using the notion of Gromov Hausdorff convergence, survey some important results in the theory of compact classes of Riemannian manifolds, and prove a new theorem concerning convergence of vector fields on sequences of manifolds.

In Section 2, we define Hausdorff and Gromov Hausdorff distance and state some basic results about these functions. In Section 3, we compare theorems in real analysis to properties of the space of all compact metric spaces under the Gromov Hausdorff metric. In Section 4, we further our discussion of compact classes through the introduction of the norm of a Riemannian manifold. In Section 5, we prove a theorem by Cheeger concerning compact classes of positively curved manifolds. In Section 6, we discuss a relevant version of the Arzela Ascoli theorem and prove a theorem about conformal Killing vector fields on convergent sequences of manifolds. This is interesting because it involves two different types of convergence, that of manifolds and that of vector fields.

2. Sequences of Metric Spaces

In order to work with the concept of convergence of metric spaces, one must define a distance function on the space of all metric spaces. In [1], Peter Petersen defines two such functions:

Hausdorff distance: Let (X, d) be a metric space, and $Y_1, Y_2 \subseteq X$ be subsets. The Hausdorff distance between Y_1 and Y_2, denoted $d_H(Y_1, Y_2)$, is given by $\inf\{\varepsilon : Y_1 \subseteq B(Y_2, \varepsilon), Y_2 \subseteq B(Y_1, \varepsilon)\}$. Here, $B(Z, \varepsilon)$ is the ball around the subset Z of X of radius ε.

Theorem 1. *d_H is an extended pseudometric on the collection of non-empty subsets of a metric space. It can be turned into an extended metric by identifying subsets with the same closure, and into a metric by insisting that the subsets are bounded.*

We then generalise this to a distance between arbitrary metric spaces:

Gromov-Hausdorff distance: Let X and Y be metric spaces. The Gromov-Hausdorff distance is defined to be the smallest Hausdorff distance between X and Y when they are both isometrically embedded as subsets of another metric space M:

Date: 8 March 2015.

$$d_{GH}(X,Y) = \inf_{M,f,g} d_H(f(X), g(Y))$$

where $f : X \to M$, $g : Y \to M$ are embeddings. We say that a sequence of metric spaces $\{M_i\}$ converges to a limit M in the Gromov-Hausdorff topology if $d_{GH}(M, M_i) \to 0$ as $i \to \infty$.

Note that it is certainly possible for $d_{GH}(X,Y) = \infty$. To prevent this, it is necessary either to restrict ourselves to the class of compact metric spaces, or else to restrict ourselves to the larger class of proper metric spaces (those for which closed balls are compact) and modify the distance function so that it is based at a certain point in the space. The resulting metric is called the pointed Gromov-Hausdorff distance and is defined as follows: let X, Y be metric spaces and $x \in X, y \in Y$.

Then

$$d_{GH}((X,x),(Y,y)) = \inf_{M,f,g}\{d_H(f(X),(Y)) + d(x,y)\}$$

where M, f, g are as before. A sequence of pointed metric spaces (M_i, m_i, d_i) converges to a limit (M, m, d) in the pointed Gromov-Hausdorff topology if for all $R > 0$, the closed metric balls $(B(m_i, R), m_i)$ converge to $(B(m, R), m)$ with respect to the metric above. It is essential that the metric spaces X and Y be proper, for otherwise the closed balls would not necessarily be compact, and so distances may not be finite. Note that this means that it is difficult to define a sensible notion of convergence for many infinite dimensional spaces.

Example 1. *It is generally difficult to calculate Gromov-Hausdorff distance. Here is an example based on* [2] *of an explicit calculation:*

Let $\mathbb{S}^n(r_1), \mathbb{S}^n(r_2) \subset \mathbb{R}^{n+1}$ *be spheres with the induced metric from* $\mathbb{R}^{n+1}$*. Then* $d_{GH}(\mathbb{S}^n(r_1), \mathbb{S}^n(r_2)) = |r_1 - r_2|$*. This is achieved by embedding the two spheres so that their centres are the same. Also note that* $diam(\mathbb{S}^n(r_i)) = 2r_i$*. We have the inequality*

$$d_{GH}(X,Y) \geq \frac{1}{2}|diam(X) - diam(Y)|$$

which implies that $d_{GH}(\mathbb{S}^n(r_1), \mathbb{S}^n(r_2)) \geq \frac{1}{2}|2r_1 - 2r_2| = |r_1 - r_2|$*, so the bound is sharp.*

Theorem 2. *Gromov-Hausdorff distance is a pseudometric on the space of compact metric spaces, and it becomes a metric if isometric spaces are identified.*

Proof. We prove that compact metric spaces X, Y with $d_{GH}(X,Y) = 0$ must be isometric. It is possible to choose a sequence of metrics on XY with $d_H(X,Y) < \frac{1}{i}$.

There are functions

$$I_i : X \to Y$$
$$J_i : Y \to X$$

with $d_i(x, I_i(x)) \leq \frac{1}{i}$ and $d_i(y, J_i(y)) \leq \frac{1}{i}$.

Using the triangle inequality, we obtain bounds for $d(I_i(x_1), I_i(x_2)), d(J_i(y_1), J_i(y_2))$, $d(x, J_i(I_i(x)))$, and $d(y, I_i(J_i(y)))$. Using a diagonal argument, it can be shown that there are functions $I : X \to Y$ and $J : Y \to X$ that are the limits of the I_i, J_i. Moreover, they are both distance decreasing and uniformly continuous. So I and J are inverses, and hence both isometries.

□

A sequence of manifolds doesn't necessarily converge in the Gromov-Hausdorff topology to another manifold, and similarly, it is possible for a sequence of metric spaces (without a manifold structure) to converge to a manifold. The following example illustrates another phenomenon that can occur in Gromov-Hausdorff convergence:

Example 2 (Dimension Collapse)**.** *Consider the sequence $\{M_i\}_{i\in\mathbb{N}}$ where $M_i = \mathbb{S}^k \times \frac{1}{i}\mathbb{S}^{n-k}$ of n-dimensional manifolds. The limit as $i \to \infty$ is $\mathbb{S}^k$.*

3. Parallels to Theorems in Real Analysis

Although Gromov Hausdorff convergence doesn't preserve some properties of elements in the sequence, such as dimension and smoothness, it is a very useful way to define the distance between arbitary metric spaces; many properties of sequences of real numbers also hold true in the Gromov-Hausdorff topology, as shown in [3].

Theorem 3. *The limit of a convergent sequence is unique up to isometry.*

This follows from theorem 2.

Theorem 4. *The space of compact metric spaces with the Gromov-Hausdorff metric is complete.*

Proof. Let $\{X_n\}$ be a Cauchy sequence of metric spaces. To prove completeness, we want to show that it has a convergent subsequence. Select a subsequence $\{X_i\}$ where $d_{GH}(X_i, X_{i+1}) < \frac{1}{2^i}$ for all i.

Choose metrics $d_{i,i+1}$ on the disjoint union of X_i and X_{i+1} that make the Hausdorff distance between them less than or equal to $\frac{1}{2^i}$.

Define a metric $d_{i,i+j}$ on the disjoint union of X_i and X_{i+j} as follows:

$$d_{i,i+j}(x_i, x_{i+j}) = \min_{x_{i+k} \in X_{i+k}} \{ \sum_{k=0}^{j-1} d_{i+k,i+k+1}(x_{i+k}, x_{i+k+1}) \}$$

Then, we can define a metric, d, on the disjoint union of the X_i such that the Hausdorff distance between X_i and X_{i+j} is less than or equal to $\frac{1}{2^{i-1}}$.

Consider the space $\hat{X} = \{\{x_i\} : x_i \in X_i, d(x_i, x_j) \to 0 \text{ as } i, j \to \infty\}$. We define a pseudometric

$$d(\{x_i\}, \{y_i\}) = \lim_{i\to\infty} d(x_i, y_i)$$

If we identify sequences that are 0 distance apart in this pseudometric, then we obtain a metric. Extend the metric on the disjoint union of the X_i to a metric on the disjoint union of the X_i and X:

$$d(x_k, \{x_i\}) = \lim_{i\to\infty} d(x_k, x_i)$$

Since the Hausdorff distance between X_j and X_{j+1} is less than or equal to 2^{-j}, we can, given $x_i \in X_i$ find a sequence $\{x_{i+j}\} \in \hat{X}$ such that $x_{i+0} = x_i$ and $d(x_{i+j}, x_{i+j+1}) \leq 2^{-j}$. So $d(x_i, \{x_{i+j}\}) \leq 2^{-i+1}$. For any given sequence $\{x_i\}$, there is an equivalent sequence $\{y_i\}$ with $d(y_k, \{y_i\}) \leq 2^{-k+1}$ for all k.

□

Theorem 5. *The space of compact metric spaces with the Gromov-Hausdorff metric is separable.*

Proof. The collection of all metric spaces with finitely many points is dense in the space of compact metric spaces, since each has an ε-dense subset. The subcollection of metric spaces with rational distances between all pairs of points is countable and dense. □

The Bolzano-Weierstrass Theorem in real analysis states that every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence. For compact classes of metric spaces, this property also holds.

Also interesting are precompact classes - those for which every sequence in the class has a subsequence that converges to a limit that may not be in the class. Gromov [4] gives the following characterisation of precompact classes:

Theorem 6. *For a class $\mathcal{C}$, the following statements are equivalent:*

(1) *$\mathcal{C}$ is precompact.*

(2) *There exists a function* $N_1(\varepsilon) : (0, \alpha) \to (0, \infty)$ *such that the maximum number of disjoint* $\frac{\varepsilon}{2}$ *balls in any element of the class is less than* $N_1(\varepsilon)$.
(3) *There exists a function* $N_2(\varepsilon) : (0, \alpha) \to (0, \infty)$ *such that the minimum number of* ε *balls used to cover any element of the class is less than* $N_2(\varepsilon)$.

It follows that a collection of pointed metric spaces is compact iff for all $R > 0$ the collection

$$\{B(x, R) \subset (X, x)\}$$

is precompact.

Example 3. *The collection of pointed complete Riemannian manifolds* (M^n, g) *with* $Ric \geq (n-1)k$ *is precompact.*

Use the relative volume comparison theorem: Fix R, and suppose that $B(x_1, \varepsilon), \ldots, B(x_l, \varepsilon) \subset B(x, R)$ *are disjoint. If* $B(x_i, \varepsilon)$ *is the ball with the smallest volume, then*

$$l \leq \frac{volB(x, r)}{volB(x_i, \varepsilon)} \leq \frac{v(n, k, 2R)}{v(n, k, \varepsilon)}$$

where $v(n, k, r)$ *is the function that denotes the volume of a ball of radius* r *in the* n*-dimensional constant curvature* k *space form. The conclusion follows from the second point above.*

This is a reasonably simple characterisation of precompact spaces, however, compact classes are generally constructed using the concept of the norm of a manifold.

4. Norm of a Manifold

We define the norm of a manifold at the scale of r to measure how much the space looks like the Euclidean ball $B(0, r)$. Euclidean space itself is the only space of norm zero, and in general, the norm at the scale of r measures how flat the space is at that scale. See [5] for more details about the way in which the norm is constructed, and an alternate way in which it can be done.

Definition 1. *Suppose* $A \subset (M, g)$. *The* $C^{m,\alpha}$ *norm on the scale of* r *of* A *is written* $||A \subset (M, g)||_{C^{m,\alpha},r}$.

We have

$$||A \subset (M, g)||_{C^{m,\alpha},r} \leq Q$$

if there are charts $\phi_s : B(0, r) \subset \mathbb{R}^n \leftrightarrow U_s \subset M$ *where*

(1) *(Lebesgue Number) Every ball* $B(p, \frac{1}{10}e^{-Q}r)$*, where* $p \in A$*, is contained in some* U_s*.*
(2) *(Bounds for Metric Coefficients) We have* $|D\phi_s| \leq e^Q$ *on* $B(0,r)$ *and* $|D\phi_s^{-1}| \leq e^Q$ *on* U_s*.*
(3) *(Bounds on Metric Derivatives) We have* $r^{|j|+\alpha}||D^j g_s||_\alpha \leq Q$ *for all multi-indices* j *with* $0 \leq |j| \leq m$*. Here* g_s *is the matrix of metric coefficients in the* ϕ_s *coordinates.*
(4) *(Variation of Bounds in Co-ordinate Overlaps)* $||\phi_s^{-1} \circ \phi_t||_{C^{m+1,\alpha}} \leq (10 + r)e^Q$*.*

The norm has many useful properties, as described in [6]:

Theorem 7.

$$||A \subset (M,g)||_{C^{m,\alpha},r} = ||A \subset (M, \lambda^2 g)||_{C^{m,\alpha},\lambda r}$$

for all $\lambda > 0$*.*

Proof. Rescale the metric $g \mapsto \lambda^2 g$, and change the domain of the charts from $B(0,r)$ to $B(0,\lambda r)$. □

Theorem 8. *The function* $r \mapsto ||A \subset (M,g)||_{C^{m,\alpha},\lambda r}$ *is continuous and converges to 0 as* $r \to 0$*.*

Proof. Modify the charts from $\phi_s : B(0,r) \to M$ to $\phi_s(\lambda^{-}1x) : B(0,\lambda r) \to M$, but do not change the metric. It can be shown that the norm is continuous in r by considering the norm on the scale of r_i where $r_i \to r$.

Locally, at the point $p \in M$, we have $g_{ij} = \delta_{ij}$. So $|D\phi|_p| = |D\phi^{-1}|_p| = 1$. These coordinates on sufficiently small balls give charts with zero norm, hence, as $r \to 0$, the norm at the scale of r also approaches zero. □

Theorem 9. *If* (M_i, p_i, g_i) *converges to* (M,p,g) *in the pointed* $C^{m,\alpha}$ *topology then for a precompact domain* $A \subset M$ *there are precompact domains* $A_i \subset M_i$ *with*

$$||A_i||_{C^{m,\alpha},r} \to ||A||_{C^{m,\alpha},r}$$

for all $r > 0$*.*

Proof. Fix r. For given A, pick a domain Ω containing A with the property that for large i, there are embeddings $f_i : \Omega \to M_i$ with $f_i^* g_i \to g_i$ on the domain Ω.

Put $A_i = f_i(A)$. It can be shown that for charts in M_i given by

$$\phi_{i,s} = f_i \circ \phi_s : B(0,r) \to M_i$$

the theorem holds.

□

Theorem 10. *For a complete flat manifold M, we have $||(M,g)||_{C^{m,\alpha},r} = 0$ for all $r \leq$ inj(M,g). The norm of Euclidean space is 0 at all scales.*

Conversely: M is a flat manifold if $||(M,g)||_{C^{m,\alpha},r} = 0$ for some r, and M is Euclidean space if $||(M,g)||_{C^{m,\alpha},r} = 0$ for all r.

Proof. The first implication is trivial.

For the converse: it can be shown that M can be covered by charts $\phi : B(0,r) \to M$ that satisfy

$$d(\phi(x_1), \phi(x_2)) \leq |x_1 - x_2|$$

$$d(\phi(x_1), \phi(x_2)) \geq \min\{|x_1 - x_2|, (2r - |x_1| - |x_2|)\}$$

It can be concluded that ϕ are locally distance preserving, and one-to-one. This implies that they are isometries.

□

The following compactness theorem makes use of the norm:

Theorem 11 (The Convergence Theorem of Riemannian Geometry)**.** *Given $i_0, K > 0$ there exist $Q, r > 0$ such that any (M,g) with inj$\geq i_0$ and $|sec| \leq K$ has $||(M,g)||_{C^1,r} \leq Q$. Furthermore, this class is compact in the pointed C^α topology for all $\alpha < 1$.*

Proof. Given a Riemannian manifold (M,g) with inj$\geq i_0$, $|\sec| \leq K$, and a point $p \in M$, the distance function $d(x) = d(x,p)$ is smooth on $B(p,i_0)$ and the Hessian is bounded in absolute value on $B(p,i_0) \setminus B(p, \frac{i_0}{2})$ by a function depending on the dimension of the manifold, K, and i_0. (See [15]). Using the fact that $g_{ij}(p) = \delta_{ij}$ locally for any $p \in M$, we can conclude that there are ε, Q depending on i_0, K for which the second condition in the definiton of the norm (bounds on metric coefficients) holds in $B(p,\varepsilon), B(0,\varepsilon)$ and for norm Q. By the inverse function theorem, it is possible to find a ball $B(0,r) \subset \mathbb{R}^n$ such that the charts satisfy the second condition. The first condition follows from applying this argument at every $p \in M$, and the third and fourth conditions follow from the Hessian estimate.

□

5. Cheeger's Theorem

Theorem 12. *For given $n \geq 1$ and $k > 0$, the class of Riemannian $2n$-manifolds with $k \leq sec \leq 1$ is compact and thus contains finitely many diffeomorphism types.*

Note that Cheeger's proof actually gives C^α convergence, which implies Gromov-Hausdorff convergence since any $C^{m,\alpha}$ convergence is stronger than Gromov-Hausdorff convergence .

Proof. The Convergence Theorem in Riemannian Geometry states that given any $i_0, K > 0$, the class of Riemannian manifolds with inj$\geq i_0$, and sec$\leq K$ is compact in the pointed Gromov-Hausdorff topology. Klingenberg's Injectivity Radius estimate [7] implies that any manifold in the class given in Cheeger's theorem is in this class.

By [9], such manifolds have bounded diameter, and Cheeger [8] implies that there exist only finitely many diffeomorphism types.

□

6. Convergence of Vector Fields on Manifolds

We require a notion of convergence for functions with different domains. Peter Petersen [10] suggests the following:

Suppose $f_k : X_k \to Y_k$, and $X_k \to X$, $Y_k \to Y$. We will say that the functions f_k converge to a function $f : X \to Y$ if for every sequence of points $x_k \in X_k$ converging to a point $x \in X$, we have $f_k(x_k) \to f(x)$.

We will need to use the Arzela-Ascoli theorem. In this context, we say that a sequence of functions is equicontinuous if for all $\varepsilon > 0$ there is a $\delta > 0$ such that $f_k(B(x_k, \delta)) \subset B(f_k(x_k), \delta)$, for all k and x_k.

Theorem 13 (Arzela-Ascoli)**.** *An equicontinuous family $\{f_k\}$ where $f_k : X_k \to Y_k$, $X_k \to X$, $Y_k \to Y$ in the Gromov-Hausdorff topology has a convergent subsequence.*

Example 4. *Any family of functions which are Lipschitz with the same Lipschitz constant is equicontinuous.*

My original theorem goes as follows:

Theorem 14. *Suppose $\{(M_k^{2n}, g_k)\}$ is a sequence of Riemannian manifolds satisfying*

(1) *$vol(M_k) \leq v$ for all k*
(2) *$0 < k \leq sec(M_k) \leq 1$ for all k*

and that $(M_k^{2n}, g_k) \to (M_\infty^{2n}, g_\infty)$. Let $\{X_k\}$ be a sequence of conformal Killing fields such that X_k is a conformal Killing field on (M_k^{2n}, g_k). That is,

$$\mathcal{L}_{X_k} g_k = \phi_k g_k$$

where $\phi_k : M_k \to \mathbb{R}$ is smooth. Then $(M_\infty^{2n}, g_\infty)$ has $0 < k \leq sec(M_k) \leq 1$ and if $\phi_k \to \phi$ for some smooth function $\phi : M_\infty \to R$ then $\{X_k\}$ has a subsequence that converges to a conformal Killing field X_∞ on $(M_\infty^{2n}, g_\infty)$. Furthermore, $(M_\infty^{2n}, g_\infty)$ is one of finitely many diffeomorphism types.

Proof. To prove the subconvergence of the sequence of conformal Killing fields, we aim to apply the Arzela-Ascoli theorem for functions on different domains. Let η_k be the 1-form dual to the vector field X_k. The following formula, due to Bochner, holds for all manifolds such that $|Ric| \leq \lambda, \text{inj} \geq i_0$, and $\text{vol} \leq v$:

$$\Delta \eta_k = 2 \sum_{i,j} Ric(\partial_i, \partial_j) \eta_k^j dx^i \tag{6.1}$$

Note that it also holds for all manifolds satisfying the conditions in the theorem since the curvature condition implies that $|Ric|$ is bounded, and Klingenberg's injectivity radius estimate gives a lower bound on $inj(M_k)$ in even dimensions. Anderson's method from [12] yields a uniform L^p bound $(1 \leq p < \infty)$ for $Ric(\partial_i, \partial_j)$, possibly depending on p, by bounding the terms of the Ricci tensor on balls around a point on the manifold which have harmonic co-ordinates satisfying certain conditions on the metric. Since (6.1) is elliptic, this gives a $C^{1,\alpha}$ bound for η_k, and hence for X_k. So $\{X_{ki}\}_{i \in \mathbb{N}}$ converges to X_∞ on $(M_\infty^{2n}, g_\infty)$.

X_∞ is a conformal Killing field on $(M_\infty^{2n}, g_\infty)$ since it satisfies $\mathcal{L}_{X_\infty} g_\infty = \phi g_\infty$.

$(M_\infty^{2n}, g_\infty)$ satisfies $0 < k \leq \sec(M_k) \leq 1$, and the class only contains finitely many diffeomorphism types due to theorem 12.

□

We then have the following result for existence of Killing fields:

Corollary 1. *Suppose that we have the same assumptions on (M_k^{2n}, g_k) and $\{X_k\}$ as in theorem 14. Then X_∞ is a Killing field if and only if $\phi \equiv 0$.*

Of course, this implies that if X_k is Killing for all k, then so is X_∞, since $\phi_k \equiv 0$ for all k. However, X_∞ is not necessarily non-trivial. Even if X_k is non-trivial for all k, it is possible for X_∞ to be trivial, such as in the following example: for all k, let (M_k^{2n}, g_k) be $\mathbb{S}^{2n}(1) = \{\mathbf{x} : x_1^2 + \cdots + x_{2n}^2 = 1\} \subset \mathbb{R}^{2n+1}$ with the induced metric, and X_k be the rotation by $\frac{\pi}{k}$ about $x_1 = 0$. As $k \to \infty$, $\frac{\pi}{k} \to 0$, so X_∞ is simply the identity.

Example 5. *In dimension 2, we can use the Gauss-Bonnet theorem to obtain a topological classification of positively curved surfaces. Since*

$$\int_M K dA = 2\pi\chi(M)$$

for a surface M with area element dA, a manifold with positive sectional curvature must have a positive Euler characteristic. By the classification theorem for surfaces (see [14] *for details), the only such examples are spheres ($\chi = 2$) and projective planes ($\chi = 1$), so theorem 14 is of little use in 2 dimensions. There are very few known examples of positively curved manifolds in higher dimensions, and none apart from spheres and projective spaces in dimension greater than 24 by* [13], *so it is possible that it is also not particularly useful in higher dimensions.*

Example 6. *The Sphere Theorem (see* [16]*) states that compact, simply connected Riemannian manifolds with sectional curvature K satisfying $0 < \frac{1}{4}K_{max} < K \leq K_{max}$ are diffeomorphic to a sphere. We may set $K_{max} = 1$ by rescaling the metric, so if $0 < \frac{1}{4} < K \leq 1$ then the manifold is a sphere. Hence, if in theorem 14 we choose $k > \frac{1}{4}$ then the only manifolds in the class are spheres, so the result is trivial.*

It has been suggested that it may be of use to view compact classes as manifolds, or as special subsets of a larger manifold, however, it is unlikely that such a structure exists since, as in the previous examples, the class may be finite (which doesn't matter for compactness, but does for the construction of a differentiable structure). There is a topology on the space of all manifolds contained in an open ball in $\mathbb{R}^n$ (see [17]), however, it does not interact well with the Gromov-Hausdorff metric topology so nothing can be said about the behaviour of compact classes as subsets of this manifold.

References

[1] Petersen, P.: *Riemannian Geometry*, Graduate Texts in Mathematics Vol. 171, (2006), 294.

[2] Burago, D., Burago, I., Ivanov, S.: *A Course in Metric Geometry*, Graduate Studies in Mathematics Vol. 33, (2001), 257.

[3] Petersen, P.: *Riemannian Geometry*, Graduate Texts in Mathematics Vol. 171, (2006), 294-297.

[4] Petersen, P.: *Riemannian Geometry*, Graduate Texts in Mathematics Vol. 171, (2006), 301.

[5] Petersen, P.: *Riemannian Geometry*, Graduate Texts in Mathematics Vol. 171, (2006), 308.

[6] Petersen, P.: *Riemannian Geometry*, Graduate Texts in Mathematics Vol. 171, (2006), 309-316.

[7] Do Carmo, M. P.: *Riemannian Geometry*, Mathematics: Theory and Applications, (1992), 276-282.

[8] Berger, M.: *A Panoramic View of Riemannian Geometry*, (2007), 616.

[9] Petersen, P.: *Riemannian Geometry*, Graduate Texts in Mathematics Vol. 171, (2006), 170.

[10] Petersen, P.: *Riemannian Geometry*, Graduate Texts in Mathematics Vol. 171, (2006), 298-299.

[11] Xai, D., Shen, Z., Wei, G.: *Negative Ricci Curvature and Isometry Group*, Duke Mathematical Journal, (1994)

[12] Anderson, M.: *Convergence and Rigidity of Manifolds Under Ricci Curvature Bounds*, Invent Math 102, (1990), 429-445.

[13] Kennard, L.: *On the Hopf Conjecture with Symmetry*, PhD Thesis, (2012), 1.
[14] Ziller, W.: *Riemannian Manifolds with Positive Sectional Curvature*, (2012), 1-2.
[15] Petersen, P.: *Riemannian Geometry*, Graduate Texts in Mathematics Vol. 171, (2006), 318-319.
[16] Do Carmo, M. P.: *Riemannian Geometry*, Mathematics: Theory and Applications, (1992), 265-291.
[17] Galatius, S., Randall-Williams, O.: *Monoids of Moduli Spaces of Manifolds*, (2009), 5-7.