ENDOWING DIFFERENTIABLE MANIFOLDS WITH RIEMANNIAN STRUCTURES

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1. Introduction

The aim of this report is to explain some of the basic definitions and theorems needed for the study of Riemannian geometry, and give improved proofs of some theorems from the first three chapters of do Carmo's *Riemannian Geometry.* The main results are the proof of Theorem 2.10, in which I show that the metric constructed in do Carmo's proof satisfies the conditions given in the definition of a Riemannian metric, and the definition of the Christoffel symbols, in which I prove that the definition given by do Carmo can be derived from the unique Levi-Cevita connection on a Riemannian manifold. I also give a more detailed proof of Remark 2.3 by showing that the topology induced on a set by a differentiable structure satisfies the definition of a topology, expand the proof of Proposition 5.3 by verifying some of the properties of the Lie bracket, present a more detailed proof of Proposition 2.2 by proving some of the results used in the text, and prove an identity used in the proof of Theorem 3.6 by using the properties of metrics and vector brackets.

2. Definitions

Definition 2.1 (Topology)**.** Let X be a set and T be a family of subsets of X. T is a topology on X if both $\emptyset$ and $X \in T$, any union of elements of T is in T, and any intersection of finitely many elements of T is in T.

Definition 2.2 (Open)**.** Let (X, T) be a topological space. U is open iff $U \in T$.

Definition 2.3 (Metric space)**.** Suppose X is a set. Endow X with a bilinear form $d : X \times X \to \mathbb{R}$ which is symmetric, positive definite, and satisfies the triangle inequality. The tuple (X, d) is called a metric space.

Definition 2.4 (ε-neighbourhood)**.** Let (X, d) be a metric space. Suppose $U \subset X$. The ε-neighbourhood $N_\varepsilon(U)$ is the set

$$N_\varepsilon(U) = \{x \in X : d(x, u) < \varepsilon \text{ for some } u \in U\}.$$

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Definition 2.5 (Open in a metric space)**.** A set U is open if for each $x \in U$ there exists an $\varepsilon > 0$ with ε depending only on x such that the ε-neighbourhood of x is in U.

Remark 2.6. Clearly with the above definition of open any *metric space* is also a *topological space*, by setting the topology T to consist precisely of all the sets satisfying Definition 2.5. We call this the *topology induced by d on X*.

Definition 2.7 (Continuous)**.** $f : X \to Y$ is continuous if for any open set $V \subseteq Y$, the set $f^{-1}(V) = \{x \in X \text{ such that } f(x) \in V\}$ is open.

Definition 2.8 (Homeomorphism)**.** A homeomorphism is a function that is bijective, continuous and has a continuous inverse.

Definition 2.9 (Smooth)**.** A smooth function is a function for which derivatives of any order exists.

2.1. **Definition of a Differentiable Manifold.** Multivariable calculus is an extension of the techniques of calculus of functions of one variable to the more general spaces $\mathbb{R}^n$. The classical motivation behind the study of differentiable manifolds is the application of calculus to more general spaces.

Definition 2.10. An n-dimensional differentiable manifold is a set M and a family of injective mappings $x_\alpha : U_\alpha \subset \mathbb{R}^n \to M$ of open sets $U_\alpha \subset \mathbb{R}^n$ into M satisfying the following conditions:

(i) $\bigcup_\alpha x_\alpha(U_\alpha) = M$. So the images of the open sets U_α under the parametrisation x_α covers M.
(ii) For any pair α, β with $x_\alpha(U_\alpha) \cap x_\beta(U_\beta) = W \neq \emptyset$ the sets ${x_\alpha}^{-1}(W)$ and ${x_\beta}^{-1}(W)$ are open sets in $\mathbb{R}^n$. The mappings ${x_\beta}^{-1} \circ x_\alpha$ are differentiable, where $\circ$ represents composition.
(iii) The family of charts $\{(U_\alpha, x_\alpha)\}$ is maximal relative to the above conditions. A differentiable structure on M can be completed to a maximal one by including the parametrisations that, along with the given ones, satisfy the above condition.

This definition of a differentiable structure induces a topology on M. A set $A \subset M$ is an open set in M iff $x_\alpha^{-1}\big(A \cap x_\alpha(U_\alpha)\big)$ is an open set in $\mathbb{R}^n$ for every α. M and the empty set are both open sets in M.

Theorem 2.11. *The union of a finite number of open sets is open.*

Proof (If M has a metric structure.) Let $I = \{1, \ldots, n\}$. The set U_i is open in M $\forall i \in I$. Let $y \in \bigcup_{i \in I} x_i(U_i)$. Then $y \in x_k(U_k)$ for some

$k \in I$. Since $x_k(U_k)$ is open in M, $\exists \epsilon > 0$: the ϵ-ball of $y \subseteq x_k(U_k) \subseteq \bigcup_{i \in I} x_i(U_i)$. □

Proof (For general M.) Suppose we are given two sets A, B which are open in M. Then

$$x_\alpha^{-1}\big(A \cap x_\alpha(U_\alpha)\big) \quad \text{and} \quad x_\alpha^{-1}\big(B \cap x_\alpha(U_\alpha)\big)$$

are open as subsets of $\mathbb{R}^n$ for every α. We need to show that

$$x_\alpha^{-1}\big((A \cup B) \cap x_\alpha(U_\alpha)\big) \tag{1}$$

is open as a subset of $\mathbb{R}^n$ for every α. Expanding this expression gives:

$$x_\alpha^{-1}\Big(\big(A \cap x_\alpha(U_\alpha)\big) \cup \big(B \cap x_\alpha(U_\alpha)\big)\Big) \tag{2}$$

By the definition of a differentiable manifold, x_α is injective. If its range is restricted to its image then x_α is also surjective, so it is a bijection. So:

$$x_\alpha^{-1}\Big((A \cup B) \cap \big(x_\alpha(U_\alpha)\big)\Big) = x_\alpha^{-1}\big(A \cap x_\alpha(U_\alpha)\big) \cup x_\alpha^{-1}\big(B \cap x_\alpha(U_\alpha)\big) \tag{3}$$

This is the union of open sets in $\mathbb{R}^n$ and so it is an open subset of $\mathbb{R}^n$.

We have thus shown (1) and so the result for countably many open sets follows by induction. □

Remark 2.12. We present also the proof where M has a metric structure with a view toward the eventual introduction of a Riemannian metric on M.

Theorem 2.13. *The intersection of finitely many open sets is open.*

Proof (If M has a metric structure). Let $y \in \bigcap_{i=1}^n x_i(U_i)$. For every $i \in \{1, \ldots, n\}$, $y \in x_i(U_i)$ So $\exists \epsilon_i > 0$: the ϵ_i-ball of $y \subseteq x_i(U_i)$. Let $\epsilon = \min(\epsilon_i)$ for $i \in \{1, \ldots, n\}$. Then the ϵ-ball of $y \subseteq \epsilon_i$-ball of $y \subseteq x_i(U_i)$. So the ball of $y \subseteq \bigcap_{i=1}^n x_i(U_i)$. □

Proof (For general M). By a similar method to the proof for the union of open sets, we need to show that

$$x_\alpha^{-1}\big((A \cap B) \cap x_\alpha(U_\alpha)\big) \tag{4}$$

is an open set in $\mathbb{R}^n$.

$$x_\alpha^{-1}\big((A \cap B) \cap x_\alpha(U_\alpha)\big) = x_\alpha^{-1}\Big(\big(A \cap x_\alpha(U_\alpha)\big) \cap \big(B \cap x_\alpha(U_\alpha)\big)\Big) \tag{5}$$

Because x_α is a bijection, this expression is given by:

$$x_\alpha^{-1}\big(A \cap x_\alpha(U_\alpha)\big) \cap x_\alpha^{-1}\big(B \cap x_\alpha(U_\alpha)\big) \tag{6}$$

This subset is open in $\mathbb{R}^n$. □

Definition 2.14. M_1^n and M_2^m are differentiable manifolds. A mapping $\phi : M_1 \to M_2$ is differentiable at $p \in M_1$ if, given a parametrisation $y : V \subset \mathbb{R}^n \to M_1$ at $\phi(p)$ there is a parametrisation $x : U \subset \mathbb{R}^n \to M_1$ at p such that:

- $\phi(x(U)) \subset y(V)$; and
- $y^{-1} \circ \phi \circ x : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x^{-1}(p)$.

It follows that ϕ is differentiable on an open set of M_1 if it is differentiable at every point $p \in M_1$.

Example 1. $\mathbb{R}^n$: Use the mapping $e : \mathbb{R}^n \to \mathbb{R}^n$. $\{(\mathbb{R}^n, e)\}$ is an atlas (but not a maximal one).

Example 2. Regular surfaces in $\mathbb{R}^3$: A subset $S \subset \mathbb{R}^3$ is a regular surface if for every point $p \in S$ there exists a neighbourhood V of p and a mapping $x : U \subset \mathbb{R}^2 \to V \cap S$ of an open set $U \subset \mathbb{R}^2$ onto $V \cap S$ such that x is a differentiable homeomorphism and dx_q (the differential of x at q) is injective for all $q \in U$.

Example 3. $\mathbb{S}^2$: One way to define an atlas is to use stereographic projection from the 2-sphere to $\mathbb{R}^2$. First, consider stereographic projection from the north pole $(0, 0, 1)$. Draw a line from the north pole through the point (x, y, z). The point $(x', y', 0)$, the intersection of this line with the plane $z = 0$, is the point to which (x, y, z) is mapped. Using the equation of a line $\frac{x_1 - x_2}{x_1 - x_3} = \frac{y_1 - y_2}{y_1 - y_3} = \frac{z_1 - z_2}{z_1 - z_3}$, substitute the north pole and the point (x, y, z) to obtain the co-ordinates of the point $(x', y', 0)$. So $n(x) = x' = \frac{x}{1-z}$ and $n(y) = y' = \frac{y}{1-z}$. However, using this projection, the north pole is not mapped to any point. This means that a second parametrisation is required. Use stereographic projection from the south pole, $(0, 0, -1)$, mapping the point (x, y, z) on $\mathbb{S}^2$ to the point $(x'', y'', 0)$ on $z = 0$. Using the equation of the line, it can be shown that $s(x) = x'' = \frac{x}{1+z}$ and $s(y) = y'' = \frac{y}{1+z}$. So $\{(\mathbb{R}^2, n), (\mathbb{R}^2, s)\}$ is an atlas.

3. Tangent Vectors

The fundamental property used in the definition of tangent vectors to differentiable manifolds is the fact that the directional derivative with respect to a vector v is an operator that depends uniquely on v. This is a suitable property to use because it is true in the case of $M = \mathbb{R}^n$:

Let $\alpha : (-\epsilon, \epsilon) \to \mathbb{R}^n$ be a differentiable curve in $\mathbb{R}^n$, where $\alpha(0) = p$.

$\alpha(t) = \big(x_1(t), \ldots, x_n(t)\big)$ and $v = \alpha'(0) = \big(x_1'(0), \ldots, x_n'(0)\big)$.

Restrict f, a differentiable function defined in a neighbourhood of p, to the curve α. The directional derivative with respect to $v \in \mathbb{R}^n$ is:

$$\frac{d(f \circ \alpha)}{dt}\bigg|_{t=0} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\bigg|_{t=0} \frac{dx_i}{dt}\bigg|_{t=0} = \sum_i x_i'(0) \frac{\partial}{\partial x_i} f$$

using the multivariate chain rule $D(G \circ F)_a = DG_b DF_a$ where $b = F(a)$.

Definition 3.1. Let M be a differentiable manifold. Let $\alpha : (-\epsilon, \epsilon) \to M$ be a differentiable curve in M. Let $\mathcal{D}$ be the set of functions on M that are differentiable at p. Suppose $\alpha(0) = p \in M$. The tangent vector to the curve α at $t = 0$ is a function $\alpha'(0) : \mathcal{D} \to \mathbb{R}$ given by:

$$\alpha'(0) f = \frac{d(f \circ \alpha)}{dt}\bigg|_{t=0}$$

with $f \in \mathcal{D}$, so the tangent vector at p is the tangent vector at $t = 0$ of a curve α for which $\alpha(0) = p$.

Definition 3.2. T_pM is the set of all tangent vectors to M at p, called the tangent space.

Consider a parametrisation $x : U \to M^n$ at $p = x(0)$.

$f \circ x(q) = f\big(x(x_1, \ldots, x_n)\big)$ with $q = (x_1, \ldots, x_n) \in U$

$x^{-1} \circ \alpha(t) = x^{-1}\big(x_1(t), \ldots, x_n(t)\big)$

So $f \circ \alpha = f \circ x \circ x^{-1} \circ \alpha$

Restrict f to α:

$\alpha'(0) f = \frac{d}{dt}(f \circ \alpha)\big|_{t=0} = \frac{d}{dt} f\big(x_1(t), \ldots, x_n(t)\big)\big|_{t=0} = \big(\sum_i x_i'(0)(\frac{\partial}{\partial x_i}_0)\big) f.$

Theorem 3.3 (Lemma)**.** *If M is an n-dimensional differentiable manifold then T_pM is an n-dimensional vector space.*

Proof. $\big((\frac{\partial}{\partial x_1})_0, \ldots, (\frac{\partial}{\partial x_n})_0\big)$ is a basis of T_pM determined by the parametrisation x. There are n vectors in this basis so T_pM is an n-dimensional vector space. □

Example 4. The tangent space at a point (x_0, y_0, z_0) of $\mathbb{S}^2$ is the tangent plane at that point. $f(x, y, z) = x^2 + y^2 + z^2 - 1$. Then

$\frac{\partial}{\partial x}(x,y,z) = 2x$, $\frac{\partial}{\partial y}(x,y,z) = 2y$, and $\frac{\partial}{\partial z}(x,y,z) = 2z$. The normal vector at the point (x_0, y_0, z_0) is given by $n = \nabla_f(x_0, y_0, z_0) = (2x_0, 2y_0, 2z_0)$. A vector is in the tangent plane iff $PM \cdot n = 0$. So $(x - x_0, y - y_0, z - z_0) \cdot (2x_0, 2y_0, 2z_0) = 0$, and the equation of the tangent plane is $2x_0(x - x_0) + 2y_0(y - y_0) + 2z_0(z - z_0) = 0$.

Theorem 3.4. *M_1^n and M_2^n are differentiable manifolds. Let $\phi : M_1 \to M_2$ be a differentiable mapping. For all $p \in M_1$ and for all $v \in T_pM_1$, choose a differentiable curve $\alpha : (-\epsilon, \epsilon) \to M$ such that $\alpha(0) = p, \alpha'(0) = v$. The mapping $d\phi_p : T_pM_1 \to T_{\phi(p)}M_2$ defined as $d\phi_p(v) = (\phi \circ \alpha)'(0)$ is a linear mapping that is independent of the choice of α.*

Proof. Let $x : U \to M_2$ be a parametrisation at p and $y : V \to M_2$ be a parametrisation at $\phi(p)$.

$y^{-1} \circ \phi \circ x(q) = y(x_1, \ldots, x_n)$ where $(x_1, \ldots, x_n) \in U$

The domain of x needs to be restricted so that the image of $\phi \circ x(q)$ is a subset of the domain of y^{-1}.

$y^{-1} \circ \phi \circ x(q) = \big(y_1(x_1, \ldots, x_n), \ldots, y_n(x_1, \ldots, x_n)\big)$

Express the curve α in x as follows: $x^{-1} \circ \alpha(t) = \big(x_1(t), \ldots, x_2(t)\big)$

Using composition of mappings, $\big(y^{-1} \circ \phi \circ x\big) \circ \big(x^{-1} \circ \alpha(t)\big) = y^{-1} \circ \phi \circ \alpha(t) = \Big(y_1\big(x_1(t), \ldots, x_n(t)\big), \ldots, y_m\big(x_1(t), \ldots v, x_n(t)\big)\Big)$.

Using the basis of the tangent space $T_{\phi(p)}M_2$ associated to y, this expression can be written as a matrix: $d\phi_p(v) = \left(\frac{\partial y_i}{\partial x_j}\right)\big(x_j'(0)\big)$ where $\frac{\partial y_i}{\partial x_j}$ is an $m \times n$ matrix (i goes from 1 to m and j goes from 1 to n), and $\big(x_j'(0)\big)$ is a column vector with n entries. So $d\phi_p$ is a linear mapping $T_pM_1 \to T_{\phi(p)}M_2$. □

This linear mapping is the differential of the mapping ϕ at p.

Definition 3.5. Let M_1 and M_2 be differentiable manifolds. A mapping $\phi : M_1 \to M_2$ is a diffeomorphism if it is differentiable, bijective and its inverse is differentiable.

Theorem 3.6. *Diffeomorphism is an equivalence relation.*

Proof. Let M be a differentiable manifold. $M \cong M$ because the identity mapping is a diffeomorphism, so it is reflexive. If $M_1 \cong M_2$ then there exists a diffeomorphism $f : M_1 \to M_2$. Because f is a bijection, its inverse $f^{-1} : M_2 \to M_1$ is a bijection. f is a diffeomorphism so f and f^{-1} are both differentiable. This shows that f^{-1} is a diffeomorphism because f^{-1} and $(f^{-1})^{-1} = f$ are differentiable, so the symmetric condition is satisfied. If $f : M_1 \to M_2$ and $g : M_2 \to M_3$ are diffeomorphisms then $g \circ f : M_1 \to M_3$ is a composition of bijections, so it is also a bijection. f and g are both differentiable so $g \circ f$ is also differentiable. Also, f^{-1} and g^{-1} are differentiable so $f^{-1} \circ g^{-1} : M_3 \to M_1$ is differentiable, thus, the transitive condition is satisfied. □

4. Vector Fields

Definition 4.1. The tangent bundle TM is the set $\{(p, v) : p \in M, v \in T_pM\}$.

Definition 4.2. A vector field X on a differentiable manifold M is a mapping $X : M \to TM$ that takes a point $p \in M$ to a vector $X(p) \in T_pM$. X is differentiable if the mapping $X : M \to TM$ is differentiable.

For any parametrisation $x : U \subset \mathbb{R}^n \to M$, $X(p) = \sum_{i=1}^{n} a_i(p) \frac{\partial}{\partial x_i}$, where a_i is a function $a_i : U \to \mathbb{R}$ and $\frac{\partial}{\partial x_i}$ is the basis of T_pM associated to the parametrisation x. X is differentiable iff $a_i, i \in \{1, \ldots, n\}$ are differentiable.

Theorem 4.3. *Let X and Y be differentiable vector fields on a differentiable manifold M. There exists a unique vector field Z that satisfies $Zf = (XY - YX)f$ for every $f \in \mathcal{D}$ where $\mathcal{D}$ is the set of differentiable functions on M.*

Proof. Assume that $\exists Z$ satisfying $Zf = (XY - YX)f$ $\forall f \in \mathcal{D}$. Let $p \in M$ and let $x : U \to M$ be a parametrisation at p. Then

$$X = \sum_i a_i \frac{\partial}{\partial x_i}, Y = \sum_j b_j \frac{\partial}{\partial x_j} .$$

For all $f \in \mathcal{D}$:

$$XYf = X\Big(\sum_j b_j \frac{\partial}{\partial x_j}\Big)f = \sum_{i,j} a_i \frac{\partial b_j}{\partial x_i} \frac{\partial f}{\partial x_j} + \sum_{i,j} a_i b_j \frac{\partial^2 f}{\partial x_i \partial x_j}$$

using the chain rule.

Also,

$$YXf = Y\Big(\sum_i a_i \frac{\partial}{\partial x_i}\Big) f = \sum_{i,j} b_j \frac{\partial a_i}{\partial x_j}\frac{\partial f}{\partial x_j} + \sum_{i,j} a_i b_j \frac{\partial^2 f}{\partial x_i \partial x_j}$$

So

$$Zf = XYf - YXf = \sum_{i,j} a_i \frac{\partial b_j}{\partial x_i}\frac{\partial f}{\partial x_i} - \sum_{i,j} b_j \frac{\partial a_i}{\partial x_j}\frac{\partial f}{\partial x_i} = \sum_{i,j}\Big(a_i \frac{\partial b_j}{\partial x_i} - b_j \frac{\partial a_j}{\partial x_i}\Big)\frac{\partial f}{\partial x_j}$$

This shows that if Z exists, it is unique. To prove existence, define Z_α in each co-ordinate neighbourhood $x_\alpha(U_\alpha)$ of a differentiable structure on M using the expansion for Z above. $Z_\alpha = Z_\beta$ on the non-empty set $x_\alpha(U_\alpha) \cap x_\beta(U_\beta)$ by uniqueness. This implies that Z can be defined over the whole of M as it is defined for each co-ordinate neighbourhood, and uniqueness shows that $Z_\alpha = Z_\beta$ for all α, β. Z is differentiable because both X and Y are. □

Definition 4.4. For vector fields X, Y, the Lie bracket $[X, Y]$ is defined as $[X, Y] = XY - YX$.

Theorem 4.5. *Let X, Y and Z be differentiable vector fields on M, and a and b be real numbers. Then:*

(i) $[X, Y] = -[Y, X]$;
(ii) $[aX + bY, Z] = a[X, Z] + b[Y, Z]$;
(iii) $\big[[X, Y], Z\big] + \big[[Y, Z], X\big] + \big[[Z, X], Y\big] = 0$.

Proof. Observe that

$$-[Y, X] = -(YX - XY) = XY - YX = [X, Y]$$

which proves (i). For (ii), we compute

$$\begin{aligned}[aX + bY, Z] &= (aX + bY)Z - Z(aX + bY)\\ &= aXZ + bYZ - aZX + bZY\\ &= a[X, Z] + b[Y, Z]\,,\end{aligned}$$

which proves (ii). Finally,

$$\begin{aligned}&\big[[X, Y], Z\big] + \big[[Y, Z], X\big] + \big[[Z, X], Y\big]\\ &\qquad = [XY - YX, Z] + [YZ - ZY, X] + [ZX - XZ, Y]\\ &\qquad = XYZ - YXZ - ZXY + ZYX + YZX - ZYX - XYZ\\ &\qquad\qquad + XZY + ZXY - XZY - YZX + YXZ\\ &\qquad = 0\,,\end{aligned}$$

which completes the proof. $\square$

Definition 4.6. A curve $\alpha : (-\delta, \delta) \to M$ such that $\alpha'(t) = X\big(\alpha(t)\big)$ and $\alpha(0) = q$ is a trajectory of the vector field X that passes through q when $t = 0$. A unique trajectory passes through each point of a neighbourhood.

Definition 4.7. $\phi_t : U \to M$ where $\phi_t(q) = \phi(t, q)$ is the local flow of the vector field X.

Theorem 4.8. *Let X, Y be differentiable vector fields on M, let $p \in M$, and let ϕ_t be the local flow of X in a neighbourhood U of p. Then*

$$[X, Y](p) = \lim_{t\to 0} \frac{1}{t}\big(Y - d\phi_t Y\big)\big(\phi_t(p)\big)$$

Theorem 4.9 (Lemma)**.** *Let $h : (-\delta, \delta) \times U \to \mathbb{R}$ be a differentiable mapping with $h(0, q) = 0 \quad \forall q \in U$. There exists a differentiable mapping $g : (-\delta, \delta) \times U \to \mathbb{R}$ such that $h(t, q) = tg(t, q)$ given by $g(0, q) = \frac{\partial h(t,q)}{\partial t}\Big|_{t=0}$.*

Proof. Define $g(t, q) = \int_0^1 \frac{\partial h(ts,q)}{\partial(ts)} ds$. Substitute $x = ts$, so $dx = tds$ and $ds = \frac{1}{t} ds$. When $s = 0, x = 0$ and when $s = 1, x = t$. Then

$$\int_0^1 \frac{\partial h(ts, q)}{\partial(ts)} ds = \frac{1}{t}\int_0^t \frac{\partial h(x, q)}{\partial(x)} dx$$

So

$$tg(t, q) = \int_0^t \frac{\partial h(ts, q)}{\partial(ts)} d(ts) = h(t, q)$$

$\square$

Proof of Theorem 4.8. Let f be a differentiable function defined in a neighbourhood of p. Set $h(t, q) = f(\phi_t(q)) - f(q)$. Now use the result above to obtain a differentiable function $g(t, q)$ satisfying

$$f \circ \phi_t(q) = f(q) + tg(t, q) \quad \text{where} \tag{7}$$

$$g(0, q) = Xf(q)\,. \tag{8}$$

Apply the chain rule for composition of functions to obtain:

$$\big((d\phi_t Y)f\big)\big(\phi_t(p)\big) = \big(Y(f \circ \phi_t)\big)(p) = Yf(p) + t\big(Yg(t, p)\big) \quad \text{using (7).}$$

Therefore

$$\begin{aligned}\lim_{t\to 0} \frac{1}{t}(Y - d\phi_t Y)f(\phi_t p) &= \lim_{t\to 0} \frac{(Yf)(\phi_t p) - Yf(p)}{t} - \big(Yg(0, p)\big) \text{ again using (7)}\\ &= \lim_{t\to 0} \frac{Y\big(f \circ \phi_t(q) - f(q)\big)}{t} - Y\big(X(f)\big)(p)\,.\end{aligned}$$

By rearranging (7) and substituting in (8) we find

$$\begin{aligned}\lim_{t\to 0}\frac{1}{t}(Y - d\phi_t Y)f(\phi_t p) &= \lim_{t\to 0}\frac{tY\big(g(0,q)\big)}{t} - Y\big(X(f)\big)(p)\\ &= \big(X(Yf)\big)(p) - \big(Y(Xf)\big)(p)\\ &= \big((XY - YX)f\big)(p)\\ &= \big([X,Y]f\big)(p)\,.\end{aligned}$$

This finishes the proof. □

5. Riemannian Metrics

In $\mathbb{R}^3$, the length of tangent vectors can be measured using the dot product: $\langle (u_1,u_2,u_3),(v_1,v_2,v_3)\rangle = u_1v_1 + u_2v_2 + u_3v_3$. An intrinsic way of measuring length of tangent vectors to a differentiable manifold that doesn't use the ambient space is required. This metric should vary differentiably between points, and also have the properties of an inner product for any u, v, w:

(i) Symmetric: $\langle u,v\rangle = \langle v,u\rangle$
(ii) Bilinear: $\langle u+v,w\rangle = \langle u,w\rangle + \langle v,w\rangle$. Similarly, $\langle u,v+w\rangle = \langle u,v\rangle + \langle u,w\rangle$. Also, for a scalar k, $\langle ku,v\rangle = \langle u,kv\rangle = k\langle u,v\rangle$
(iii) Positive Definite: $\langle u,u\rangle \geq 0$ and $\langle u,u\rangle = 0$ iff $u = 0$.

Definition 5.1. A Riemannian metric on a differentiable manifold M associates an inner product $\langle\cdot,\cdot\rangle_p$ on the tangent space T_pM to each point $p \in M$. This inner product is symmetric, bilinear and positive definite, and varies differentiably as follows:

If $x : U \subset \mathbb{R}^n \to M$ is a system of co-ordinates around p with $x(x_1,\dots,x_n) = q \in x(U)$ and $\frac{\partial}{\partial x_i}(q) = dx_q(0,\dots,1,\dots,0)$ then $\left\langle \frac{\partial}{\partial x_i}(q), \frac{\partial}{\partial x_j}(q)\right\rangle_q = g_{ij}(x_1,\dots,x_n)$ where g_{ij} is the local representation of the Riemannian metric in the parametrisation x is a differentiable function on U.

Remark 5.2. g_{ij} is a $(0,2)$ tensor field on M.

Definition 5.3. A differentiable manifold M with a given Riemannian metric g is called a Riemannian manifold, and written as a pair (M,g).

The concept of isometry is used to determine whether two Riemannian manifolds are the same.

Definition 5.4. Let (M, g) and (N, h) be Riemannian manifolds. A diffeomorphism $f : M \to N$ is an isometry if

$$g(u, v)_p = h(df_{f(p)}(u), df_{f(p)}(v))_{f(p)} \text{ for all } p \in M, \forall u, v \in T_pM.$$

Example 5. Riemannian Metric: Standard Euclidean space: Let $\frac{\partial}{\partial x_i} = e_i$, where e_i is the unit vector with a 1 in the i^{th} position. Then $\langle e_i, e_j \rangle = \delta_{ij}$ where $\delta_{ij} = 1$ if $i = j$, 0 otherwise.

Example 6. Riemannian Metric: A Lie group is a group G with a differentiable structure where the maps $G \times G \to G : (x, y) \to xy$ and $G \to G : x \to x^{-1}$ are smooth.

Example 7. Lie group: $\mathbb{R}^n$ is a differentiable manifold. Under the binary operation addition, it satisfies all the group axioms. So $\mathbb{R}^n$ is a Lie group.

Theorem 5.5. *The mappings $L_x : y \longmapsto xy$ and $R_x : y \longmapsto yx$ are diffeomorphisms.*

Proof. L_x is a bijection because $x_1 y = x_2 y$ implies that $x_1 = x_2$ (multiply on the right by y^{-1}), and every $g \in G$ is the image of an element $x^{-1}g$. L_x is the restriction of $G \times G \to G$ to $\{x\} \times G \to G$, so it is differentiable. ${L_x}^{-1}(y) = x^{-1}y = L_{x^{-1}}(y)$ so its inverse is also differentiable. By a similar argument, R_x is also a diffeomorphism. □

Definition 5.6. A Riemannian metric on G is left invariant if

$$g(u, v)_y = h(d(L_x)_y u, d(L_x)_y v)_{L_{x(y)}}$$

for all $x, y \in G, u, v \in T_yG$.

Theorem 5.7. *Let X and Y be left invariant vector fields. Then $[X, Y]$ is a left invariant vector field.*

Proof.

$$\begin{aligned} dL_x[X, Y]f &= [X, Y](f \circ L_x) \quad \text{using the definition of } dL_x \\ &= X(dL_x Y)f - Y(dL_x X)f \\ &\quad \text{using the definition of Lie bracket } [X, Y]f = X\big(Y(f)\big) - Y\big(X(f)\big) \\ &= \Big(X\big(Y(f)\big) - Y\big(X(f)\big)\Big) \quad \text{using the definition of } L_x = [X, Y]f \end{aligned}$$

□

To each vector $X_eG \in T_eG$, associate the vector field X satisfying $X_a = dL_a X_e$, for $a \in G$. Define $[X_e, Y_e] = [X, Y]_e$ for $X_e, Y_e \in T_eG$.

A left invariant Riemannian metric on G can be defined by taking any inner product $\langle\cdot,\cdot\rangle$ on the tangent space T_eG and then define $\langle u, v\rangle_x$, for $x \in G$ and $u, v \in T_xG$:

$$g(u,v)_x = h\Big(\big(dL_{x^{-1}}\big)_x(u), \big(dL_{x^{-1}}\big)_x(v)\Big)_e$$

Definition 5.8. Let M^m and N^n be differentiable manifolds. A differentiable mapping $\phi : M \to N$ is an immersion if the differential of ϕ at p $d\phi_p : T_pM \to T_{\phi(p)}N$ is injective $\forall p \in M$.

Example 8. Immersion: The curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ given by $\alpha(t) = (t^2 - 1, t + 2)$ is an immersion. The necessary condition for this is that $\alpha'(t) \neq 0$, which is true because $\alpha'(t) = (2t, 1) \neq 0$ for any $t \in \mathbb{R}$

Example 9. Riemannian metric: Construction of a metric on immersed manifolds: Let $f : M \to N$ be an immersion. If there is a Riemannian metric on N, then f induces a Riemannian metric on M: define $\langle u, v\rangle_p = \langle df_p(u), df_p(v)\rangle_{f(p)}$ for $u, v \in T_pM$.

Definition 5.9. A metric space X is Hausdorff if, given two distinct points of X, there exist neighbourhoods of these points that do not intersect.

Example 10. $\mathbb{R}$ with the metric $d(x, y) = |y - x|$

Definition 5.10. A manifold M has a countable basis if M can be covered by a countable number of co-ordinate neighbourhoods.

Example 11. $\mathbb{R}$ with the metric $d(x, y) = |y - x|$ where the basis is the set of subsets of $\mathbb{R}^n$: (a, b) with $a, b \in \mathbb{Q}$.

Theorem 5.11. *A differentiable manifold that is Hausdorff and has a countable basis has a Riemannian metric.*

To prove this theorem, the concept of a differentiable partition of unity is required:

Definition 5.12. A family of open sets $V_\alpha \subset M$ with $\bigcup_\alpha V_\alpha = M$ is locally finite if every point $p \in M$ has a neighbourhood W such that $W \cap V_\alpha \neq \emptyset$ for only a finite number of indices α.

Definition 5.13. The closure of a set S is the set of all points x such that for every $k > 0$ there exists $y \in S$ such that $\langle x, y\rangle < k$ where $\langle\cdot,\cdot\rangle$ is a metric on the space.

Definition 5.14. The support of a function $f : M \to (R)$ is the closure of the set of points where $f \neq 0$.

Definition 5.15. A family f_α of differentiable functions $f_\alpha : M \to \mathbb{R}$ is a differentiable partition of unity subordinate to the open cover $\{V_\alpha\}$ if:

(i) $\forall\ \alpha, f_\alpha \geq 0$ and the support of f_α is contained in a co-ordinate neighbourhood $C_\alpha \subset V_\alpha$;
(ii) The family V_α is locally finite;
(iii) $\sum_\alpha f_\alpha(p) = 1$ for all $p \in M$.

Example 12. Defining a metric on a patch of $\mathbb{S}^2$.

Consider the parametrisation $f(\theta,\psi) = (\sin(\theta)\cos(\psi), \cos(\theta)\sin(\psi), \cos(\theta))$. Because $f(\theta,\psi)\cdot f(\theta,\psi) = 1$, this gives the 2-sphere.

$$\begin{aligned}
\frac{\partial f(\theta,\psi)}{\partial\theta} &= (\cos(\theta)\cos(\psi), \cos(\theta)\sin(\psi), -\sin(\theta)) \\
\frac{\partial f(\theta,\psi)}{\partial\psi} &= (-\sin(\theta)\sin(\psi), \sin(\theta)\cos(\psi), 0) \\
f_{\theta\psi} = f_{\psi\theta} &= -\cos(\theta)\cos(\psi)\sin(\theta)\sin(\psi) + \sin(\theta)\cos(\theta)\sin(\psi)\cos(\psi) = 0 \\
f_{\theta\theta} &= \cos^2(\theta)\cos^2(\psi) + \cos^2(\theta)\sin^2(\psi) + \sin^2(\theta) = 1 \\
f_{\psi\psi} &= \sin^2(\theta)\sin^2(\psi) + \sin^2(\theta)\cos^2(\psi) = \sin^2(\theta)
\end{aligned}$$

So $g_{\theta\psi} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2(\theta) \end{pmatrix}$

Proof of Theorem. Let f_α be a differentiable partition of unity on M subordinate to the covering V_α. Define a Riemannian metric $\langle,\rangle^\alpha$ on each open set V_α as the metric induced by the system of local co-ordinates. Set $\langle u,v\rangle_p = \sum_\alpha f_\alpha(p)\langle u,v\rangle_p^\alpha$ for all $p \in M, u, v \in T_pM$. To show that $\langle u,v\rangle_p$ is a Riemannian metric, the conditions from the definition must be verified:

(i) Symmetric: $\langle\cdot,\cdot\rangle^\alpha$ is symmetric. $\langle u,v\rangle_p^\alpha = \langle v,u\rangle_p^\alpha$. Using symmetry, $\sum_\alpha f_\alpha(p)\langle u,v\rangle_p^\alpha = \sum_\alpha f_\alpha(p)\langle v,u\rangle_p^\alpha$ so $\langle u,v\rangle_p = \langle v,u\rangle_p$ and the metric is symmetric.

(ii) Bilinear: Use the bilinearity of $\langle , \rangle^\alpha$.

$$\begin{aligned}\langle u+v, w\rangle_p^\alpha &= \sum_\alpha f_\alpha(p)\langle u+v, w\rangle_p \\ &= \sum_\alpha f_\alpha(p)\big(\langle u, w\rangle_p^\alpha + \langle v, w\rangle_p^\alpha\big) \\ &= \sum_\alpha f_\alpha(p)\langle u, w\rangle_p^\alpha + \sum_\alpha f_\alpha(p)\langle v, w\rangle_p^\alpha \\ &= \langle u, w\rangle_p + \langle v, w\rangle_p.\end{aligned}$$

Therefore the metric is bilinear.

(iii) Positive Definite: $f_\alpha \geq 0$ and $\langle , \rangle^\alpha \geq 0$ so $\sum_\alpha f_\alpha(p)\langle u, v\rangle_p^\alpha \geq 0$ therefore $\langle u, v\rangle_p$ is positive definite.

(iv) Metric varies differentiably between points: This is true for the metric $\langle , \rangle^\alpha$. The sum and product of differentiable functions is a differentiable function, so $\sum_\alpha f_\alpha(p)\langle u, v\rangle_p^\alpha$ is a differentiable function.

Therefore $\langle u, v\rangle_p$ is a Riemannian metric. □

6. Affine Connections

The notion of covariant derivative is needed in order to differentiate vector fields. It is a generalisation of the directional derivative.

Definition 6.1. Let M be a differentiable manifold, let $\mathcal{X}(M)$ denote the set of smooth vector fields on M and let $\mathcal{D}(M)$ denote the ring of smooth functions defined on M. An affine connection ∇ on M is a mapping $\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ given by $\nabla : (X, Y) \to \nabla_X Y$. ∇ has these properties for $X, Y, Z \in \mathcal{X}(M)$, $f, g \in \mathcal{D}(M)$:

(i) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$
(ii) $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$
(iii) $\nabla_X(fY) = f\nabla_X Y + X(f)Y$

Let $X = \sum_{i=1}^n x_i X_i$, $Y = \sum_{j=1}^n y_j X_j$ where $X_i = \frac{\partial}{\partial x_i}$ is the basis determined by a parametrisation x.

$\nabla_X Y = \sum_i x_i \nabla_{X_i}(\sum_j y_j X_j) = \sum_{i,j} x_i y_j \nabla_{X_i} X_j + \sum_{i,j} x_i X_i y_i X_j$.

Let $\nabla_{X_i} X_j = \sum_k \Gamma_{ij}^k X_k$. This implies that Γ_{ij}^k are differentiable. The functions Γ_{ij}^k are called the Christoffel symbols.

Theorem 6.2. *Let M be a differentiable manifold equipped with an affine connection ∇. There exists a unique covariant derivative that maps a vector field V along a differentiable curve $c : I \to M$ to another vector field $\frac{DV}{dt}$ along c. The covariant derivative satisfies the following:*

(i) $\frac{D}{dt}(V + W) = \frac{DV}{dt} + \frac{DW}{dt}$

(ii) $\frac{D}{dt}(fV) = \frac{df}{dt}V + f\frac{DV}{dt}$ *where W is a vector field along c and f is a differentiable function on I.*

(iii) *If $V(t) = Y\big(c(t)\big)$ (V is induced by the vector field Y) then $\frac{DV}{dt} = \nabla_{\frac{dc}{dt}}Y$*

Proof. To prove uniqueness, assume that the covariant derivative exists and satisfies the properties given in the definition. Let $x : U \in \mathbb{R}^n$ be a parametrisation such that $c(I) \cap x(U) \neq \emptyset$. Let $\big(x_1(t), x_2(t), \dots, x_n(t)\big)$ be the local expression of $c(t)$, with $t \in I$, and $X_i = \frac{\partial}{\partial x_i}$. Express V locally as the field $V = \sum_j v_j(t)X_j\big(c(t)\big)$ where $j \in \{1, \dots n\}$.

Using the second property:

$$\begin{aligned}\frac{DV}{dt} &= \Big(\frac{dv_1(t)}{dt}X_1\big(c(t)\big) + v_1(t)\frac{Dx_1\big(c(t)\big)}{dt}\Big) + \dots + \Big(\frac{dv_n(t)}{dt}X_n\big(c(t)\big) + v_n(t)\frac{Dx_n\big(c(t)\big)}{dt}\Big)\\ &= \Big(\frac{dv_1(t)}{dt}X_1\big(c(t)\big) + \dots \frac{dv_n(t)}{dt}X_n\big(c(t)\big)\Big) + \Big(v_1(t)\frac{Dx_1(c(t))}{dt} + \dots v_n(t)\frac{Dx_n\big(c(t)\big)}{dt}\Big)\\ &\text{by re-grouping the terms}\\ &= \sum_{j=1}^{n}\frac{dv_j}{dt}X_j + \sum_{j=1}^{n}v_j\frac{DX_j}{dt}.\end{aligned}$$

□

Theorem 6.3 (Lemma)**.**

$$\frac{dc}{dt} = \sum_{i=1}^{n}\frac{dx_i}{dt}\frac{\partial c(t)}{\partial x_i}$$

Proof.

$$c(t) = \big(x_1(t), \dots, x_n(t)\big)$$

$$\begin{aligned}\frac{dc}{dt} &= c'(t) = \big(x_1'(t), \dots, x_n'(t)\big)\\ &= \big(x_1'(t), 0, \dots, 0\big) + \dots + \big(0, \dots, x_n'(t)\big)\\ &= x_1'(t)\frac{\partial c(t)}{\partial x_1} + \dots + x_n'(t)\frac{\partial c(t)}{\partial x_n}\\ &= \sum_{i=1}^{n} x_i'(t)\frac{\partial c(t)}{\partial x_i}\end{aligned}$$

□

Using the third property:

$$\frac{DX_j}{dt} = \nabla_{\sum_i \frac{dx_i}{dt}X_i} X_j = \sum_i \frac{dx_i}{dt}\nabla_{X_i}X_j$$

by the definition of an affine connection

Therefore

$$\frac{DV}{dt} = \sum_j \frac{dv_j}{dt}X_j + \sum_{i,j}\frac{dx_i}{dt}v_j\Gamma_{ij} \tag{9}$$

So this correspondence is unique.

To prove that such a mapping exists, define $\frac{DV}{dt}$ in $x(U)$ by (9). If $y(W)$ is another parametrisation, such that $y(W) \cap x(U) \neq \emptyset$ and $\frac{DV}{dt}$ is defined in $y(W)$ by (9) then the definitions of $\frac{DV}{dt}$ agree in in $y(W) \cap x(U)$, by the uniqueness of $\frac{DV}{dt}$ in $x(U)$. So this definition can be extended over the whole of M, and this shows that $\frac{DV}{dt}$ exists.

Definition 6.4. Let M be a differentiable manifold, V be a vector field along a curve $c : I \to M$ and ∇ be an affine connection on M. V is called *parallel* when $\frac{DV}{dt} = 0\ \forall t \in I$.

Theorem 6.5. *Let M be a differentiable manifold with an affine connection ∇, let $c : I \to M$ be a differentiable curve in M and let V_0 be a vector tangent to M at $c(t_0)$ where $t_0 \in I$ (V_0 is in the tangent space $T_{c(t_0)}M$). There exists a unique parallel vector field $V = \sum_j v_j X_j$ where $X_j = \frac{\partial}{\partial x_j}(c(t))$ along c such that $V(t_0) = V_0$.*

Proof. Let $x^{-1}(c(t)) = (x_1(t), \ldots, x_n(t))$ be the local expression of $c(t)$ and let $V_0 = \sum_j v_{j_0} X_j$. Suppose there exists a vector field $V \in x(U)$ which is parallel along c, and which satisfies $V(t_0) = V_0$.

Then V satisfies $\frac{DV}{dt} = 0$. Using the previous theorem,

$$\sum_j \frac{dv_j}{dt} X_j + \sum_{i,j} \frac{dx_i}{dt} v_j \nabla_{X_i} X_j = 0$$

Substitute $\nabla_{X_i} X_j = \sum_k \Gamma_{ij}^k X_k$ and relabel the indices:

$$\sum_k \left(\frac{dv_k}{dt} + \sum_{i,j} v_j \frac{dx_i}{dt} \Gamma_{i,j}^k \right) X_k = 0$$

Consider the system of differential equations given by:

$$\frac{dv_k}{dt} + \sum_{i,j} v_j \frac{dx_i}{dt} \Gamma_{i,j}^k \quad \text{for} \quad k = 1, \ldots, n$$

By the Cauchy-Kovalevski theorem, it has a unique solution that satisfies the initial condition $v_k(t_0) = v_k(0)$. So if a solution V exists, then it is unique. Because the system is linear, there is a solution for all $t \in I$. This shows that the theorem is true when $c(I)$ is in a co-ordinate neighbourhood $x(U)$ of $x : U \subset \mathbb{R}^n \to M$.

For any $t_1 \in I$ the segment $c\big([t_0, t_1]\big) \subset M$ can be covered by a finite number of co-ordinate neighbourhoods due to compactness. V is defined in each of these co-ordinate neighbourhoods, and by uniqueness, the definitions of V are the same when the intersections of the co-ordinate neighbourhoods are non-empty. So V can be defined along $[t_0, t_1]$. $V(t)$ is the parallel transport of $V(t_0)$ along the curve c. □

Definition 6.6. M is a differentiable manifold with an affine connection ∇ and Riemannian metric $\langle , \rangle$. A connection is compatible with the metric if $\langle P, P' \rangle$ is a constant for any smooth curve c and any pair of parallel vector fields P and P' along c.

Theorem 6.7. *For a Riemannian manifold M, a connection ∇ on M is compatible with a metric iff for any vector fields V, W along the differentiable curve $c : I \to M$: $\frac{d}{dt}\langle V, W \rangle = \langle \frac{DV}{dt}, W \rangle + \langle V, \frac{DW}{dt} \rangle$ where $t \in I$.*

Proof. $\Leftarrow$ direction:

$\frac{d}{dt}\langle V, W\rangle = \langle \frac{DV}{dt}, W\rangle + \langle V, \frac{DW}{dt}\rangle$ is true for parallel vector fields P, P', so:

$\frac{d}{dt}\langle P, P'\rangle = \langle \frac{DP}{dt}, P'\rangle + \langle P, \frac{DP'}{dt}\rangle$
But $\frac{DP}{dt} = 0$ and $\frac{DP'}{dt} = 0$
So $\frac{d}{dt}\langle P, P'\rangle = \langle 0, P'\rangle + \langle P, 0\rangle = \langle 0X, P'\rangle + \langle P, 0Y\rangle$ for some X, Y
Therefore $\frac{d}{dt}\langle P, P'\rangle = 0\langle X, P'\rangle + 0\langle P, Y\rangle = 0$

So ∇ is compatible with the metric.

$\Rightarrow$ direction: Choose an orthonormal basis $\{P_1(t_0), \ldots, P_n(t_0)\}$ of the tangent space $T_{x(t_0)}M$, where $t_0 \in I$. Using the previous theorem, the vectors $P_i(t_0)$ can be extended along c to $P_i(t)$ using parallel transport. ∇ is compatible with the metric on M so $\{P_1(t), \ldots, P_n(t)\}$ is an orthonormal basis of $T_{c(t)}M$.

$$V = \sum_i v_i P_i, \ \ W = \sum_i w_i P_i$$

where $i \in \{1, \ldots, n\}$, and v_i, w_i are differentiable functions on I.

$$\text{So } \frac{DV}{dt} = \sum_i \frac{dv_i}{dt} P_i \tag{10}$$

Also,

$$\frac{DW}{dt} = \sum_i \frac{dw_i}{dt} P_i \tag{11}$$

So

$$\begin{aligned}\langle \frac{DV}{dt}, W\rangle + \langle V, \frac{DW}{dt}\rangle &= \sum_i w_i \frac{dv_i}{dt} + \sum_i \frac{dw_i}{dt} v_i \\ &= \sum_i \frac{dv_i}{dt} w_i + \frac{dw_i}{dt} v_i \\ &= \frac{d}{dt} \sum_i v_i w_i \text{using the product rule} \\ &= \frac{d}{dt} \langle V, W\rangle\end{aligned}$$

□

Theorem 6.8. *An affine connection ∇ on a Riemannian manifold M is compatible with the metric iff $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$.*

Proof. $\Leftarrow$ direction: This follows from the proof that $\frac{d}{dt}\langle V, W\rangle = \langle \frac{DV}{dt}, W\rangle + \langle V, \frac{DW}{dt}\rangle$

$\Rightarrow$ direction Let $p \in M$, $c : I \to M$ be a differentiable curve such that $c(t_0) = p$ for $t_0 \in I$, and $\frac{dc}{dt}\big|_{t=0} = X(p)$.

$$X(p)\langle Y, Z\rangle = \frac{d}{dt}\langle Y, Z\rangle\big|_{t=0} = \langle \nabla_{X(p)} Y, Z\rangle_p + \langle Y, \nabla_{X(p)} Z\rangle_p$$

This is true for all p, so $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$. □

Definition 6.9. ∇ is symmetric when the following condition is satisfied for all $X, Y \in X(M)$: $\nabla_X Y - \nabla_Y X = [X, Y]$

Theorem 6.10. *In a co-ordinate system (U, x), if ∇ is symmetric then $\Gamma^n_{ij} = \Gamma^n_{ji}$.*

Proof. Using the equality of mixed partials, $\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}$. So $X_i X_j - X_j X_i = 0$ for all i, j between 1 and n, and $[X_i, X_j] = 0$. Using the symmetric property, $\nabla_{X_i} X_j - \nabla_{X_j} X_i = [X_i, X_j] = 0$ therefore $\Gamma^n_{ij} = \Gamma^n_{ji}$. □

Theorem 6.11. *Let M be a Riemannian manifold. There exists a unique affine connection ∇ on M such that:*

(i) ∇ *is symmetric*
(ii) ∇ *is compatible with the Riemannian metric*

Theorem 6.12 (Lemma)**.**

$X\big\langle Y,Z\big\rangle + Y\big\langle Z,X\big\rangle - Z\big\langle X,Y\big\rangle = \big\langle [X,Z],Y\big\rangle + \big\langle [Y,Z],X\big\rangle + \big\langle [X,Y],Z\big\rangle + 2\big\langle Z,\nabla_Y X\big\rangle$

Proof of Lemma.

$$\begin{aligned}
\big\langle [X,Z],Y\big\rangle &+ \big\langle [Y,Z],X\big\rangle + \big\langle [X,Y],Z\big\rangle + 2\big\langle Z,\nabla_Y X\big\rangle \\
&= \langle \nabla_X Z - \nabla_Z X, Y\rangle + \langle \nabla_Y Z - \nabla_Z Y, X\rangle + \langle \nabla_X Y - \nabla_Y X, Z\rangle + 2\langle Z,\nabla_Y X\rangle \\
&= \langle \nabla_X Z, Y\rangle - \langle \nabla_Z X, Y\rangle + \langle \nabla_Y Z, X\rangle - \langle \nabla_Z Y, X\rangle + \langle \nabla_X Y, Z\rangle - \langle \nabla_Y X, Z\rangle \\
&\qquad + 2\langle Z,\nabla_Y X\rangle \\
&= \big(\langle \nabla_X Y, Z\rangle + \langle \nabla_X Z, Y\rangle\big) + \big(\langle \nabla_Y Z, X\rangle + \langle Z,\nabla_Y X\rangle\big) - \big(\langle \nabla_Z X, Y\rangle + \langle \nabla_Z Y, X\rangle\big) \\
&\qquad - \langle \nabla_Y X, Z\rangle + \langle Z,\nabla_Y X\rangle \text{ because of the symmetry of the metric} \\
&= X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle
\end{aligned}$$

□

Proof of Theorem. To prove uniqueness, assume that ∇ exists. Then:

(1) $X\langle Y,Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$

(2) $Y\langle Z,X\rangle = \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle$

(3) $Z\langle X,Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle$

Consider (1) + (2) - (3):

$X\big\langle Y,Z\big\rangle + Y\big\langle Z,X\big\rangle - Z\big\langle X,Y\big\rangle = \big\langle [X,Z],Y\big\rangle + \big\langle [Y,Z],X\big\rangle + \big\langle [X,Y],Z\big\rangle + 2\big\langle Z,\nabla_Y X\big\rangle$ using the lemma

Therefore $\langle Z,\nabla_Y X\rangle = \frac{1}{2}\Big(X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle - \big(\big\langle [X,Z],Y\big\rangle + \big\langle [Y,Z],X\big\rangle + \big\langle [X,Y],Z\big\rangle\big)\Big)$

So ∇ is determined uniquely by the choice of metric. To prove existence, define ∇ by the expression above. □

Consider this expression for $X = X_i$, $Y = X_j$ and $Z = X_k$. Using the symmetry of Γ^k_{ij}, $[X_i, X_k] = 0$ so $[X,Z] = 0$.

Similarly, $[Y,Z] = 0$ and $[X,Y] = 0$.

So $\langle Z,\nabla_Y X\rangle = \frac{1}{2}\big(X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle\big)$. Substitute the definitions of X, Y and Z from above, and note that $\langle X_i, X_j\rangle = g_{ij} = g_{ji}$:

$X\langle Y, Z\rangle = \frac{\partial}{\partial x_i} g_{jk}$

$Y\langle Z, X\rangle = \frac{\partial}{\partial x_j} g_{ki}$

$Z\langle X, Y\rangle = \frac{\partial}{\partial x_k} g_{ij}$

Substitute these into the expression for $\langle Z, \nabla_Y X\rangle$:

$\langle Z, \nabla_Y X\rangle = \frac{1}{2}(\frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ki} - \frac{\partial}{\partial x_k} g_{ij})$

Consider the term $\nabla_Y X$. Using $\nabla_{X_i} X_j = \sum_k \Gamma_{ij}^k X_k$, we obtain $\nabla_Y X = \sum_l \Gamma_{ij}^l X_l$

$$\begin{aligned}
\text{So} \quad \langle Z, \nabla_Y X\rangle &= \langle X_k, \sum_l \Gamma_{ij}^l X_l\rangle \\
&= \sum_l \langle X_k, \Gamma_{ij}^l X_l\rangle \quad \text{using bilinearity of the metric} \\
&= \sum_l \Gamma_{ij}^l \langle X_k, X_l\rangle \quad \text{using bilinearity} \\
&= \sum_l \Gamma_{ij}^l g_{kl}
\end{aligned}$$

So $\sum_l \Gamma_{ij}^l g_{kl} = \frac{1}{2}(\frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ki} - \frac{\partial}{\partial x_k} g_{ij})$

7. Conclusion

This report has set up the concepts needed for further study of Riemannian geometry. It has also presented the proofs of a few basic theorems in differential geometry, most notably, those of the theorems that every differentiable manifold has a Riemannian metric, and that on a Riemannian manifold there exists a unique symmetric affine connection compatible with the metric.