THE GELFAND-NAIMARK THEOREM

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1. Introduction

The aim of this project was to learn some basic definitions and results involving C^*-algebras, and to understand the proof of the Gelfand Naimark theorem. In this report, we follow the proofs given in [1].

We start by reviewing some basic definitions and examples involving C^*-algebras in Sections 2 and 3. In Section 4 we define the spectrum and prove the spectral radius formula. In Section 5 we define the maximal ideal space and prove that is compact. In Section 6 we define the Gelfand transform and prove that it is a homomorphism. In Section 7 we state and prove the Stone-Weierstrass theorem. Finally, in Section 8 we state and prove the Gelfand-Naimark theorem.

2. C*-Algebras

In this section we review the definition of a C^*-algebra.

Definition 1. *An* ***algebra*** *A over $\mathbb{C}$ is a vector space A over $\mathbb{C}$ with a multiplication $(a, b) \to ab$ which satisfies:*

- *Associativity: $a(bc) = (ab)c$ for all $a, b, c \in A$*
- *Distributive Law for addition: $a(b+c) = ab+ac$ for all $a, b, c \in A$*
- *Distributive Law for scalar multiplication: $z(ab) = (za)b = a(zb)$ for all $a, b \in A, z \in \mathbb{C}$*

The motivation behind the definition of an algebra is to introduce the operation of multiplication to a vector space in such a way that it is compatible with the existing operations.

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Definition 2. *A* ***Banach algebra*** *is a an algebra A over $\mathbb{C}$ which is also a Banach space (complete normed vector space) and satisfies $||ab|| \leq ||a||||b||$ for all $a, b \in A$.*

The Banach algebra identity is motivated by the desire to make multiplication continuous.

Definition 3. *A* ****-algebra*** *is an algebra A over $\mathbb{C}$ with an involution $^* : A \to A$ satisfying $(c_1 a + c_2 b)^* = \overline{c_1} a^* + \overline{c_2} b^*, (ab)^* = b^* a^*$ and $(a^*)^* = a$ for all $a, b \in A, c_1, c_2 \in \mathbb{C}$.*

The idea behind the definition of an isometry is to generalise conjugation of complex numbers. We want it to be compatible with addition, multiplication and scalar multiplication.

Definition 4. *A* ***C*-algebra*** *A is a Banach *-algebra satisfying $||a^* a|| = ||a||^2$ for all $a \in A$.*

The C*-identity has several useful consequences, such as involution becoming an isometry, and having $||1|| = 1$.

Definition 5. *A C*-algebra A is* ***commutative*** *if $ab = ba$ for all $a, b \in A$.*

3. Examples

In this section we look at several examples of C*-algebras.

Example 1. *The complex numbers*

Note that $\mathbb{C}$ is an algebra over $\mathbb{C}$. The norm is given by $|a + bi| = \sqrt{a^2 + b^2}$ and involution given by complex conjugation. It is well known that $\mathbb{C}$ is complete, and hence a Banach space. To show it is a Banach algebra, we check the Banach algebra identity. For $a + bi, c + di \in \mathbb{C}$:

$$|(a + bi)(c + di)| = |(ac - bd) + i(bc + ad)|$$

$$= \sqrt{(ac - bd)^2 + (bc + ad)^2}$$

$$= \sqrt{a^2c^2 - 2abcd + b^2d^2 + b^2c^2 + a^2d^2 + 2abcd}$$

$$= \sqrt{a^2c^2 + b^2d^2 + b^2c^2 + a^2d^2}$$

$= \sqrt{a^2+b^2}\sqrt{c^2+d^2}$

$= |a+bi||c+di|$

as required.

To prove that it is an involution: Fix vectors $a, b \in \mathbb{C}$ *and scalars* $c, d \in \mathbb{C}$. *We check:*

- $(ca+db)^* = \overline{ca+db} = \overline{ca} + \overline{db} = \overline{c}a^* + \overline{d}b^*$
- $(ab)^* = \overline{ab} = \overline{a}\overline{b} = a^*b^*$
- $(a^*)^* = \overline{\overline{a}} = a$

Finally, we verify the C^* *identity: Fix* $a \in \mathbb{C}$.

$$|a^*a| = |\overline{a}a| = |a|^2$$

as required. $\mathbb{C}$ *is a commutative* C^**-algebra.*

Example 2. $B(H)$, *the set of bounded linear operators on a Hilbert space* H

We say that a linear operator $T : X \to Y$ *is* ***bounded*** *if there exists some* $M > 0$ *such that for all* $v \in X$, *we have* $||Tv||_Y \leq M||v||_X$.

We define the operator norm for a bounded linear operator $T : H \to H$ *as*

$||T||_{op} = sup\{||Th|| : h \in H, ||h|| < 1\}$

For every bounded linear operator $T : H \to H$, *there is a unique bounded linear operator* T^* *satisfying*

$$\langle T(h), k\rangle = \langle h, T^*(k)\rangle$$

for all $h, k \in H$.

The space $B(H)$ *with norm given by the operator norm and involution given by the adjoint operation is a* C^**-algebra.*

Example 3. $M_n(\mathbb{C})$, *the set of* $n \times n$ *matrices with entries in* $\mathbb{C}$

$M_n(\mathbb{C}) = B(\mathbb{C}^n)$, that is, the set of bounded linear operators on $\mathbb{C}^n$. Hence, this space is a C-algebra as a consequence of the previous example.*

Example 4. *$C(X)$, the space of continuous functions on compact Hausdorff space X*

Here we use the sup-norm: $||f||_\infty = sup\{|f(x)| : x \in X\}$ and involution given by pointwise complex conjugation, $f^*(x) = \overline{f(x)}$. The norm is always finite since X is compact and f is continuous.

$C(X)$ is an algebra since linear combinations and products of continuous functions are also continuous. It is a Banach algebra since it is complete, and the necessary norm identity is proved as follows:

$$||fg|| = sup|f(x)g(x)| \leq (sup|f(x)|)(sup|g(x)|)$$

since $(sup|f(x)|)(sup|g(x)|)$ is an upper bound for $|f(x)g(x)|$ (and is hence greater than or equal to its supremum). So $||fg|| \leq ||f||||g||$.

We check that pointwise conjugation is an involution:

For constants $c, d \in \mathbb{C}$, we have $(cf + dg)^*(x) = \overline{(cf + dg)(x)} = \overline{c}\overline{f(x)} + \overline{d}\overline{g(x)} = \overline{c}f^*(x) + \overline{d}g^*(x)$. Now, $(fg)^*(x) = \overline{fg(x)} = \overline{f(x)g(x)} = f^*(x)g^*(x)$. Finally, $(f^*)^*(x) = \overline{\overline{f(x)}} = f(x)$, so all the properties are satisfied.

To show that $C(X)$ is indeed a C-algebra, we must verify the C* identity:*

$||f^*f|| = sup_{x \in X}|\overline{f(x)}f(x)| = sup(|f(x)|^2) = (sup|f(x)|^2) = ||f||^2$ as required.

4. The Spectrum and the Spectral Radius

In this section we define the spectrum of an element of a Banach algebra and prove the spectral radius formula.

Definition 6. *Let A be a unital Banach algebra. An element $a \in A$ is **invertible** if there exists $b \in A$ with $ab = ba = 1$. We say that b is the **inverse** of a.*

Definition 7. *Let A be a unital Banach algebra. Let $a \in A$. The* ***spectrum*** *of a is $\sigma_A(a) := \{\lambda \in \mathbb{C} : a - \lambda 1 \text{ is not invertible}\}$.*

Theorem 1. *Let A be a unital Banach algebra. Let $a \in A$. Then $\lim_{n\to\infty} ||a^n||^{\frac{1}{n}}$ exists and is equal to $r(a) := max\{|\lambda| : \lambda \in \sigma(a)\}$, the spectral radius of a.*

Proof. To show that $\lim_{n\to\infty} ||a^n||^{\frac{1}{n}} = r(a)$, we prove firstly that $r(a) \leq ||a^n||^{\frac{1}{n}}$ for all $n \in \mathbb{N}$, and secondly that for every $\epsilon > 0$, we have $||a^n||^{\frac{1}{n}} < r(a) + \epsilon$ for sufficiently large n.

For the first statement: fix $\lambda \in \mathbb{C}$. We have

$$(a^n - \lambda^n 1) = (a - \lambda 1)(a^{n-1} + \lambda a^{n-2} + \cdots + \lambda^{n-1} 1)$$

So if $a^n - \lambda^n 1$ is invertible then $a - \lambda 1$ is also. From the above equation, we have

$$(a-\lambda 1)[(a^{n-1}+\cdots+\lambda^{n-1}1)(a^n-\lambda^n 1)^{-1}] = (a^n-\lambda^n 1)(a^n-\lambda^n 1)^{-1}) = 1$$

so $[(a^{n-1} + \cdots + \lambda^{n-1}1)(a^n - \lambda^n 1)^{-1}]$ is a right inverse for $a - \lambda 1$. Similarly, $(a^n - \lambda^n 1)^{-1}(a^{n-1} + \cdots + \lambda^{n-1}1)$ is a left inverse for $a - \lambda 1$. These inverses must be equal (to a left and right inverse for $a - \lambda 1$), so we have

$$\lambda \in \sigma(a) \Leftrightarrow (a - \lambda 1) \text{ is not invertible in A}$$

$$\Rightarrow \lambda^n \in \sigma(a^n)$$

Since $\sigma(a)$ is a non-empty, closed, bounded subset of $\mathbb{C}$ contained in the closed ball $\overline{B}(0, ||a||)$ (see theorem 7 in [1]), we have $|\lambda^n| \leq ||a^n||$ for all $\lambda \in \sigma(a)$. Choose λ such that $|\lambda| = r(a)$, giving $r(a)^n \leq ||a^n||$.

For the second statement: Fix $\phi \in A^*$. Consider $f(\lambda) = \phi((a - \lambda 1)^{-1})$. We know that f is analytic on $\mathbb{C} \setminus \sigma(a)$ (again by Theorem 7 in [1]). So f is analytic for $|\lambda| > r(a)$, and it has a Laurent expansion in this region. If $|\lambda| > ||a||$ (which is greater than $r(a)$), we have

$$f(\lambda) = \phi(-\tfrac{1}{\lambda}(1 - \tfrac{a}{\lambda})^{-1} = \phi(-\tfrac{1}{\lambda}\textstyle\sum (\tfrac{a}{\lambda})^n) = -\tfrac{1}{\lambda}\sum \tfrac{\phi(a^n)}{\lambda^n}$$

Note that since $||\frac{a}{\lambda}|| < 1$, the series converges for all $|\lambda| > ||a||$, so must be the (unique) Laurent expansion for f on $|\lambda| > ||a||$. Since any function has a unique Laurent series expanion in any particular annulus, this must be the Laurent expansion for f on $|\lambda| > r(a)$.

Fix λ, where $|\lambda| > r(a)$. For all $\phi \in A^*$, there exists a constant M_ϕ depending on ϕ such that

$$|\phi(a^n)\lambda^{-n}| \leq M_\phi \ \forall n$$

So, using the uniform boundedness principle, $\{\psi_n : \phi \mapsto \phi(a^n)\lambda^{-n}\}$, a family of linear functionals on A^*, satisfies $||\psi_n|| \leq M$ for all n, where M is some constant.

The Hahn-Banach theorem gives that $||\phi_n|| = ||a^n\lambda^{-n}|| = ||a^n|||\lambda|^{-n}$. Hence, $||a^n|| \leq M|\lambda^n|$.

So, for any fixed $|\lambda| > r(a)$, there is an M such that $||a^n||^{\frac{1}{n}} \leq M^{\frac{1}{n}}|\lambda|$. As $n \to \infty$, we have $M^{\frac{1}{n}} \to 1$. Choose λ such that $r(a) < |\lambda| < r(a)+\epsilon$. Then $||a^n||^{\frac{1}{n}} \leq M^{\frac{1}{n}}|\lambda| < r(a) + \epsilon$ for sufficiently large n. This proves the second statement.

□

5. Ideals

Let A be a commutative, unital Banach algebra, and Δ_A be the maximal ideal space of A. It can be shown that Δ_A is the space of non-zero homomorphisms from A to $\mathbb{C}$ (see [1]) since maximal ideals of A correspond to the kernels of the non-zero homomorphisms $\phi : A \to \mathbb{C}$.

Definition 8. *Let (X, τ) be a topological vector space. The **weak-* topology** on X^* is the coarsest topology in which every $x \in X$ corresponds to a continuous map on X^*.*

Theorem 2. *For any commutative, unital Banach algebra A, Δ is compact in the weak-* topology on A^*.*

Proof. Alaoglu's theorem (see [1]) states that $\{\phi \in X^* : ||\phi|| \leq 1\}$ is compact in the weak* topology. We have $\Delta \subset \{\phi \in X^* : ||\phi|| \leq 1\}$, so to prove that Δ is compact, we simply need to show that it is closed.

We have

$$\Delta = \{\phi \in X^* : ||\phi|| \leq 1 \quad \text{and} \quad \phi(ab) - \phi(a)\phi(b) = 0 \forall a, b \in A\}$$
$$= \cap_{a,b \in A} \{\phi \in X^* : ||\phi|| \leq 1 \quad \text{and} \quad \phi(ab) - \phi(a)\phi(b) = 0\}$$

We define $q_{a,b}(\phi) = \phi(ab) - \phi(a)\phi(b)$. So:

$$\Delta = \cap_{a,b \in A} q_{a,b}^{-1}(0) \cap \{\phi \in A^* : ||\phi|| \leq 1\}$$

We know that a compact set in a Hausdorff space is closed, so $\{\phi \in A^* : ||\phi|| \leq 1\}$ is closed. By the continuity of the $q_{a,b}$, the set $q_{a,b}^{-1}(0)$ is closed. So Δ is the finite intersection of closed sets, and is therefore also closed.

□

6. The Gelfand Transform

Definition 9. *Let A be a commutative, unital Banach algebra. Let Δ be its maximal ideal space. The* ***Gelfand transform*** *of $a \in A$ is the function $\widehat{a} : \Delta \to \mathbb{C}$ defined by $\widehat{a}(\phi) = \phi(a)$.*

Theorem 3. *Let A be a commutative, unital Banach algebra. Let Δ be its maximal ideal space. Then the Gelfand transform is a norm-decreasing, unital homomorphism from A to $C(\Delta)$.*

Proof. To prove that $\widehat{a}$ is continuous, fix $a \in A$ and suppose $\phi_n \to \phi$ in Δ. Then

$$\widehat{a}(\phi_n) = \phi_n(a) \to \phi(a) = \widehat{a}(\phi)$$

So $\widehat{a}$ is continuous.

The map $a \to \widehat{a}$ is linear:

$$c_1\widehat{a + c_2}b(\phi) = \phi(c_1 a + c_2 b) = c_1\phi(a) + c_2\phi(b) = c_1\widehat{a}(\phi) + c_2\widehat{b}(\phi)$$

This uses the linearity of ϕ. Similarly, $a \to \widehat{a}$ is multiplicative:

$$\widehat{ab}(\phi) = \phi(ab) = \phi(a)\phi(b) = \widehat{a}(\phi)\widehat{b}(\phi)$$

We have $\phi(1_A) = 1$ for all $\phi \in \Delta$. So $\widehat{1_A}(\phi) = \phi(1_A) = 1 = 1_{C(\Delta)}(\phi)$.

Now:

$$||\widehat{a}||_\infty = sup_{\phi\in\Delta}|\widehat{a}(\phi)| = sup_{\phi\in\Delta}|\phi(a)| \leq sup_{\phi\in\Delta}||\phi||||a|| = ||a||$$

since $||\phi|| = 1$. Hence, the Gelfand transform is norm-decreasing.

□

Theorem 4. *Let A be a commutative, unital Banach algebra. Then $a \in A$ is invertible iff $\widehat{a} \in C(\Delta)$ is invertible.*

Proof. If a is invertible in A then $\widehat{a^{-1}}\widehat{a} = \widehat{a^{-1}a} = \widehat{1} = 1$ so $\widehat{a^{-1}}$ is the inverse of $\widehat{a}$, and is invertible in $C(\Delta)$.

Conversely, suppose a is not invertible in A. $J := \{ab : b \in A\}$ is a proper ideal in $A : 1 \notin J$ since $1 \in J$ implies that a is invertible. So J is contained in a maximal ideal $M = ker(\phi)$ for some $\phi \in \Delta$ (see Propositions 13 and 14 in [1]). Finally, $J \subset ker(\phi)$ implies that $a \in ker(\phi)$, so $\widehat{a}(\phi) = \phi(a) = 0$, and $\widehat{a}$ is not invertible in $C(\Delta)$. □

Theorem 5. *The spectrum of $a \in A$ is equal to the range of $\widehat{a}$, and $||\widehat{a}||_\infty = r(a)$, the spectral radius.*

Proof. Let $\lambda \in \mathbb{C}$. Then

$\lambda \notin \sigma(a) \Leftrightarrow \lambda 1 - a$ invertible in A

$\Leftrightarrow \widehat{\lambda 1 - a}$ is invertible in $C(\Delta)$

$\Leftrightarrow \lambda\widehat{1} - \widehat{a}$ is invertible in $C(\Delta)$

$\Leftrightarrow \lambda\widehat{1}(\phi) - \widehat{a}(\phi) \neq 0$ for all $\phi \in \Delta$.

$\Leftrightarrow \lambda$ is not in the range of $\widehat{a}$

So $\sigma(a) = range(\widehat{a})$, and

$r(a) = sup\{|\lambda| : \lambda \in \sigma(a) = range(\widehat{a})\}$

$= sup\{|\widehat{a}(\phi)| : \phi \in \Delta\}$

$= ||\widehat{a}||_\infty$

□

7. The Stone Weierstrass Theorem

Theorem 6. *Let X be a compact Hausdorff space. If A is a C^*-subalgebra of $C(X)$ with an identity which satisfies the property that for every $x \neq y \in X$ there is $a \in A$ with $a(x) \neq a(y)$ then $A = C(X)$.*

We follow the proof given in [2]*. Firstly, we state two definitions:*

Definition 10. *A vector subspace $A \subset C(X)$ is called a **function lattice** if $f, g \in A \Rightarrow max\{f, g\}, min\{f, g\} \in A$.*

Definition 11. *$A \subset C(X)$ **separates points strongly** if for $x, y \in X$ and $a, b \in \mathbb{R}$, there exists $f \in A$ such that $f(x) = a, f(y) = b$.*

We will use the following lemmas:

Lemma 1. *If $A \subset C(X)$ separates points and contains the constant functions, then it separates points strongly.*

Proof. Fix $x, y \in X$. Find $g \in A$ such that $g(x) = \widetilde{a}, g(y) = \widetilde{b}$, where $\widetilde{a} \neq \widetilde{b}$. Then $h = \frac{g - \widetilde{a}}{\widetilde{b} - \widetilde{a}}$ satisfies $h(x) = 0$ and $h(y) = 1$, so $f = (b-a)h+a$ satisfies $f(x) = a, f(y) = b$, and hence, A separates points strongly. □

Lemma 2. *For any subalgabra $A \subset C(X)$, its uniform closure (the set of all functions which are limits of uniformly convergent sequences of members of A) is a function lattice.*

Proof. Fix $\epsilon > 0$. The function $t \mapsto (\epsilon^2 + t)^{\frac{1}{2}}$ has a power series expansion that converges uniformly on $[0, 1]$, so we can find a polynomial $p(t)$ such that $|(\epsilon^2 + t)^{\frac{1}{2}} - p(t)| < \epsilon$ for all $t \in [0, 1]$. When $t = 0$, we have $|p(0)| < 2\epsilon$. Define $q(t) := p(t) - p(0)$. We have $q(f) \in A$ for any $f \in A$ - if $||f||_\infty \leq 1$ then

$$
\begin{aligned}
||q(f^2) - |f|||_\infty &= sup_{x\in X}|q(f^2(x)) - f^2(x)^{\frac{1}{2}}| \\
&\leq sup_{t\in[0,1]}|p(t) - p(0) - t^{\frac{1}{2}}| \\
&\leq 2\epsilon + sup_{t\in[0,1]}|p(t) - t^{\frac{1}{2}}| \\
&\leq 3\epsilon + sup_{t\in[0,1]}|(\epsilon^2 + t)^{\frac{1}{2}} - t^{\frac{1}{2}}| \\
&\leq 4\epsilon
\end{aligned}
$$

$|f| \in \bar{A}$ since $q(f^2) \in A$.

We have $max\{f, g\} = (f+g+|f-g|)$ and $min\{f, g\} = (f+g-|f-g|)$, so $max\{f, g\}$ and $min\{f, g\}$ are also in $\bar{A}$.

□

Lemma 3. *Let A be a function lattice which separates points strongly. Then A is uniformly dense in $C(X)$.*

Proof. Fix $\epsilon > 0, f \in C(X)$. The aim is to find $f_\epsilon \in A$ with $||f - f_\epsilon||_\infty < \epsilon$. Since A separates points strongly, we can find $f_{xy} \in A$ with $f_{xy}(x) = f(x), f_{xy}(y) = f(y)$.

Let $U_{xy} := \{z \in X : f(z) < f_{xy}(z) + \epsilon\}, V_{xy} := \{z \in X : f_{xy}(z) - \epsilon < f(z)\}$. These are both open sets with $x, y \in U_{xy} \cap V_{xy}$.

Fix x. As y varies, U_{xy} cover X. Since X is compact, there exists a finite subcover $U_{xy_i}, i = 1, ..., n$. Let $f_x := max f_{xy_i}$. We have $f_x \in A$ since A is a function lattice. Also, $f(z) < f_x(z) + \epsilon$ for all $z \in X$. Let $W_x = \cap V_{xy_i}$. W_x is an open neighbourhood of x and $f_x(z) < f(z)+\epsilon$ for all $z \in W_x$. The sets $\{W_x\}$ cover X, so there is exists a finite subcover $\{W_{x_i}\}, i = 1, ..., m$. Let $f_\epsilon = min\{f_{x_i}\} \in A$. We have $f(z) < f_\epsilon(z) + \epsilon$ for all $z \in X$, and additionally, $f_\epsilon(z) < f(z) + \epsilon$ for all $z \in X$. This gives the result.

□

Proof. The proof of the theorem is as follows: using the first lemma, A separates points strongly. $\bar{A}$ also separates points strongly, so, using the second lemma, it is a function lattice. By the last lemma, $\bar{A}$ is uniformly dense in $C(X)$.

□

8. The Gelfand-Naimark Theorem

In this section we prove the Gelfand-Naimark theorem.

Theorem 7. *If A is a commutative, unital C*-algebra, and Δ is the space of non-zero homomorphisms from A to $\mathbb{C}$ then $A \cong C(\Delta)$, and the isomorphism is given by the Gelfand transform.*

Lemma 4. *If $f : X \to Y$ is an isometry between two metric spaces X, Y then f is injective.*

Proof. If $f(x) \neq f(y)$ then $||f(x) - f(y)|| > 0$ so $||x - y|| > 0$, and hence $x \neq y$. □

The proof of the theorem follows:

Proof. We aim to show that the Gelfand transform is an isometric *-isomorphism between A and $C(\Delta)$, that is, that for all $a \in A$:

***-preserving**: $\widehat{a^*} = (\widehat{a})^*$

We have $\widehat{a^*}(\phi) = \phi(a^*) = \overline{\phi(a)} = (\widehat{a})^*$. The only equality that needs explanation is the second, that is, $\phi(a^*) = \overline{\phi(a)}$. The proof is as follows.

Every element of A can be written in the form $b + ic$ where b, c are both self-adjoint elements of A: set

$$b = \tfrac{a+a^*}{2}, c = -i\left(\tfrac{a-a^*}{2}\right)$$

Now, fix $\phi \in \Delta$. If $\phi(b), \phi(c) \in \mathbb{R}$ then

$$\phi(a^*) = \phi((b + ic)^*) = \phi(b^* - ic^*) = \phi(b - ic) = \phi(b) - i\phi(c) = \overline{\phi(b) + i\phi(c)} = \overline{\phi(b + ic)} = \overline{\phi(a)}$$

So we just need to show that $\phi(d)$ is real for self-adjoint d. Suppose $\phi(d) = x + iy$ for some $x, y \in \mathbb{R}$. Let $l_t = d + (it)1$ where $t \in \mathbb{R}$. We have

$|\phi(l_t)|^2 = |\phi(d) + it\phi(1)|^2$

$= |x + i(y + t)|^2$ since $\phi(1) = 1$

$= x^2 + (y + t)^2$

Now, $||\phi||_{op} = 1$ so $|\phi(l_t)|^2 \leq ||l_t||^2 = ||l_t^* l_t|| = ||(d - (it)1(d + (it)1)|| = ||d^2 + (t^2)1|| \leq ||d^2|| + t^2$ by the triangle inequality.

Hence, $x^2 + (y + t)^2 \leq ||d^2|| + t^2$, and so $x^2 + y^2 + 2yt \leq ||d^2||$ for all $t \in \mathbb{R}$. This is not true unless $y = 0$ so we must have $\phi(d) = x \in \mathbb{R}$.

Isometry: $||a||_A = ||\widehat{a}||_\infty$

We compute

$||\widehat{a}||_\infty = r(a)$ using the spectral radius formula and theorem 5

$= \lim_{n\to\infty} ||a^n||^{\frac{1}{n}}$

$= \lim_{n\to\infty} ||a^{2^n}||^{\frac{1}{2^n}}$ using the spectral radius formula

For a self-adjoint element a, we have $||a^2|| = ||a^* a|| = ||a||^2$. Using induction, we conclude that $||a^{2^n}|| = ||a||^{2^n}$.

So $||a^{2^n}||^{\frac{1}{2^n}} = ||a||$, and we have $||\widehat{a}||_\infty = \lim_{n\to\infty} ||a|| = ||a||$.

Now, suppose that $a \in A$ is arbitrary. We have

$||a||^2 = ||a^* a||$ since A is a C*-algebra

$= ||\widehat{a^* a}||$ using the argument above since $a^* a$ is self-adjoint

$= ||(\widehat{a}^* \widehat{a}||$ since the Gelfand transform is multiplicative

$= ||\widehat{a}||^2$ since $C(\Delta)$ is a C*-algebra

Surjectivity

Note that $B = \{\widehat{a} : a \in A\}$ is a sub-algebra of $C(\Delta)$: the Gelfand transform is a unital homomorphism, so $1 \in B$, and since it is an isometry (we just proved this), B is closed since A is a Banach space. If $\phi \neq \psi \in \Delta$, there must be some $a \in A$ such that $\phi(a) \neq \psi(a)$, hence,

$\widehat{a}(\phi) \neq \widehat{a}(\psi)$. So B separates points of Δ, and by the Stone-Weierstrass theorem, we have $B = C(\Delta)$. Hence, we have surjectivity.

□

References

[1] Raeburn, I.: *C*-Algebras*, (2008).
[2] Morrison, S.: *The Stone Weierstrass Theorem*, https://tqft.net/web/teaching/2014/Analysis3/LectureNotes/11-2.Stone-Weierstrass.pdf